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Fractional Simpson-like Inequalities with Parameter for Differential *s*-*tgs*-Convex Functions

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Abstract: In this paper, we first prove a new parameterized identity. Based on this identity we establish some parametrized Simpson-like type symmetric inequalities, for functions whose first derivatives are *s*-*tgs*-convex via Reimann–Liouville frational operators. Some special cases are discussed. Applications to numerical quadrature are provided.

Keywords: Newton-Cotes quadrature; *s*-*tgs*-convex functions; *P*-functions; Hölder inequality

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Citation: Merad, M.; Meftah, B.; Boulares, H.; Moumen, A.; Bouye, M. Fractional Simpson-like Inequalities with Parameter for Differential *s*-*tgs*-Convex Functions. *Fractal Fract.* **2023**, *7*, 772. https://doi.org/10.3390/fractalfract7110772

Academic Editors: Wenying Feng and Ivanka Stamova

Received: 4 July 2023
Revised: 19 October 2023
Accepted: 19 October 2023
Published: 24 October 2023

1. Introduction

Symmetric inequalities often arise in various branches of mathematics, such as algebra, analysis, and optimization. They have numerous applications and play a crucial role in proving theorems and solving problems in areas like number theory, combinatorics, and inequalities themselves. In the study of symmetric inequalities, techniques such as rearrangement inequality, Cauchy–Schwarz inequality, and the method of Lagrange multipliers are commonly employed to establish the validity of the inequalities and find optimal solutions [1–3].

Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders and their applications in different fields of sciences and engineering. In real life, fractional calculus is generated from various fractional operators such as Riemann–Liouville, Caputo, Hadamard, and so on; due to its widespread use in different fields, this calculus has attracted many researchers. The most-used operator is that of Riemann-Liouville given by the following definition

Definition 1 ([4])**.** *For any integrable function $\mathcal{L}$ on $[k,g]$ with $k \geq 0$, $I^{\varsigma}_{k^+}\mathcal{L}$ and $I^{\varsigma}_{g^-}\mathcal{L}$ are the Riemann–Liouville fractional integrals of order $\varsigma > 0$ given by*

$$\begin{aligned} I^{\varsigma}_{k^+}\mathcal{L}(x) &= \frac{1}{\Gamma(\varsigma)}\int_k^x (x-\Lambda)^{\varsigma-1}\mathcal{L}(\Lambda)d\Lambda, \quad x>k, \\ I^{\varsigma}_{g^-}\mathcal{L}(x) &= \frac{1}{\Gamma(\varsigma)}\int_x^g (\Lambda-x)^{\varsigma-1}\mathcal{L}(\Lambda)d\Lambda, \quad g>x, \end{aligned}$$

respectively, where $\Gamma(\varsigma) = \int_0^\infty e^{-\Lambda}\Lambda^{\varsigma-1}d\Lambda$ is the gamma function and

$$I^{0}_{k^+}\mathcal{L}(x) = I^{0}_{g^-}\mathcal{L}(x) = \mathcal{L}(x).$$

The theory of convexity plays a central and attractive role in many fields of research. This theory provides us with a powerful tool for solving a large class of problems that arise in pure and applied mathematics, defined as follows:

Definition 2 ([5])**.** *A function $\mathcal{L} : I \to \mathbb{R}$ is said to be convex, if*

$$\mathcal{L}(\Lambda x + (1-\Lambda)y) \leq \Lambda\mathcal{L}(x) + (1-\Lambda)\mathcal{L}(y)$$

holds for all $x, y \in I$ and all $\Lambda \in [0,1]$.

In [6], Awan et al. introduced the class of *s-tgs*-convex functions.

Definition 3 ([6])**.** *We say that a function $\mathcal{L} : I \subset \mathbb{R} \to \mathbb{R}$ is s-tgs-convex on I, if*

$$\mathcal{L}(\Lambda x + (1-\Lambda)y) \leq \Lambda^s(1-\Lambda)^s(\mathcal{L}(x) + \mathcal{L}(y))$$

holds for all $x, y \in I$ and $\Lambda \in [0,1]$, with $s \in [0,1]$.

Convexity has a close relation in the development of the theory of inequalities, of which it plays an important role in the study of qualitative properties of solutions of ordinary, partial, and integral differential equations as well as in numerical analysis, which is used for establishing the estimates of the errors for quadrature rules; see [7–18].

The following Newton–Cotes inequality involving four points is known in the literature as the 3/8-Simpson inequality

$$\left|\frac{1}{8}\left(\mathcal{L}(k) + 3\mathcal{L}\left(\frac{2k+g}{3}\right) + 3\mathcal{L}\left(\frac{k+2g}{3}\right) + \mathcal{L}(g)\right) - \frac{1}{g-k}\int_k^g \mathcal{L}(w)dw\right| \leq \frac{(g-k)^4}{6480}\left\|\mathcal{L}^{(4)}\right\|_\infty, \tag{1}$$

where $\mathcal{L}$ is a four-times continuously differentiable function on $[k, g]$ and

$$\left\|\mathcal{L}^{(4)}\right\|_\infty = \sup_{x\in[k,g]}\left|\mathcal{L}^{(4)}(x)\right|.$$

Recently, Mahmoudi and Meftah [19] discussed more general inequalities of four points and gave the following results

Theorem 1. *Let $\mathcal{L} : [k, g] \to \mathbb{R}$ be a differentiable function on $[k, g]$ such that $\mathcal{L}' \in L^1[k, g]$ with $0 \leq k < g$. If $|\mathcal{L}'|$ is s-convex in the second sense for some fixed $s \in (0,1]$, then we have*

$$\begin{aligned}
&\left|\frac{1}{2+2\rho}\left(\mathcal{L}(k) + \rho\mathcal{L}\left(\frac{2k+g}{3}\right) + \rho\mathcal{L}\left(\frac{k+2g}{3}\right) + \mathcal{L}(g)\right) - \frac{1}{g-k}\int_k^g \mathcal{L}(w)dw\right| \\
\leq\ & \frac{g-k}{9(s+1)(s+2)}\left(\left(\frac{3s+4-2\rho}{2+2\rho} + 2\left(\frac{2\rho-1}{2+2\rho}\right)^{s+2}\right)(|\mathcal{L}'(k)| + |\mathcal{L}'(g)|)\right. \\
&\left. + \left(\frac{3\rho s+(2\rho-4)}{2+2\rho} + \left(\frac{1}{2}\right)^{s+1} + 2\left(\frac{3}{2+2\rho}\right)^{s+2}\right)\left(\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right| + \left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|\right)\right),
\end{aligned}$$

where ρ is a positive number.

Theorem 2. *Let $\mathcal{L} : [k, g] \to \mathbb{R}$ be a differentiable function on $[k, g]$ such that $\mathcal{L}' \in L^1[k, g]$ with $0 \le k < g$. If $|\mathcal{L}'|^q$ is s-convex in the second sense for some fixed $s \in (0, 1]$, then we have*

$$\left|\frac{1}{2+2\rho}\left(\mathcal{L}(k) + \rho\mathcal{L}\left(\frac{2k+g}{3}\right) + \rho\mathcal{L}\left(\frac{k+2g}{3}\right) + \mathcal{L}(g)\right) - \frac{1}{g-k}\int_k^g \mathcal{L}(w)dw\right|$$
$$\le \frac{g-k}{18(p+1)^{p+1}}\left(\left(\frac{3^{p+1}+(2\rho-1)^{p+1}}{2(1+\rho)^{p+1}}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{L}'(k)|^q + \left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\frac{\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|^q + \left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}} + \left(\frac{3^{p+1}+(2\rho-1)^{p+1}}{2(1+\rho)^{p+1}}\right)^{\frac{1}{p}}\left(\frac{\left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|^q + |\mathcal{L}'(g)|^q}{s+1}\right)^{\frac{1}{q}}\right),$$

where ρ is a positive number and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 3. *Let $\mathcal{L} : [k, g] \to \mathbb{R}$ be a differentiable function on $[k, g]$ such that $\mathcal{L}' \in L^1[k, g]$ with $0 \le k < g$. If $|\mathcal{L}'|^q$ is s-convex in the second sense for some fixed $s \in (0, 1]$, then we have*

$$\left|\frac{1}{2+2\rho}\left(\mathcal{L}(k) + \rho\mathcal{L}\left(\frac{2k+g}{3}\right) + \rho\mathcal{L}\left(\frac{k+2g}{3}\right) + \mathcal{L}(g)\right) - \frac{1}{g-k}\int_k^g \mathcal{L}(w)dw\right|$$
$$\le \frac{g-k}{9((s+1)(s+2))^{\frac{1}{q}}}\left(\left(\frac{9+(2\rho-1)^2}{8(1+\rho)^2}\right)^{1-\frac{1}{q}}\left(\left(\frac{3s+4-2\rho}{2+2\rho} + 2\left(\frac{2\rho-1}{2+2\rho}\right)^{s+2}\right)|\mathcal{L}'(k)|^q\right.\right.$$
$$\left. + \left(\frac{(2\rho-1)s+(2\rho-4)}{2+2\rho} + 2\left(\frac{3}{2+2\rho}\right)^{s+2}\right)\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|^q\right)^{\frac{1}{q}}$$
$$+\frac{1}{4}\left(2s + \left(\frac{1}{2}\right)^{s-1}\right)^{\frac{1}{q}}\left(\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|^q + \left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|^q\right)^{\frac{1}{q}}$$
$$+\left(\frac{9+(2\rho-1)^2}{8(1+\rho)^2}\right)^{1-\frac{1}{q}}\left(\left(\frac{(2\rho-1)s+(2\rho-4)}{2+2\rho} + 2\left(\frac{3}{2+2\rho}\right)^{s+2}\right)\left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|^q\right.$$
$$\left.\left. + \left(\frac{3s+4-2\rho}{2+2\rho} + 2\left(\frac{2\rho-1}{2+2\rho}\right)^{s+2}\right)|\mathcal{L}'(g)|^q\right)^{\frac{1}{q}}\right),$$

where ρ is a positive number and $q \ge 1$.

Motivated by the above results, in this paper we first prove a new parameterized identity. Based on this identity, we establish some new fractional Simpson-like type inequalities for functions whose first derivatives are *s-tgs*-convex. We end this work with some applications.

2. Main Results

Let us first recall some special functions (see [4]).

The incomplete beta function is given by

$$B_m(\xi_1, \xi_2) = \int_0^m \Lambda^{\xi_1 - 1}(1-\Lambda)^{\xi_2 - 1} d\Lambda,$$

where $\xi_1, \xi_2 \in \mathbb{C}$ such that $\Re(\xi_1) > 0$, $\Re(\xi_2) > 0$ and $0 \le m < 1$. The case where $m = 1$ gives the classical beta function, i.e.,

$$B(\xi_1, \xi_2) = \int_0^1 \Lambda^{\xi_1 - 1}(1-\Lambda)^{\xi_2 - 1} d\Lambda.$$

The hypergeometric function is defined for $\Re(c) > \Re(b) > 0$ and $|z| < 1$, as follows

$$ {}_2F_1(a,b,c;z) = \frac{1}{B(b,c-b)} \int_0^1 \Lambda^{b-1}(1-\Lambda)^{c-b-1}(1-z\Lambda)^{-a} d\Lambda, $$

where $B(\cdot,\cdot)$ is the beta function.

Lemma 1. *Let $\mathcal{L} : [k,g] \to \mathbb{R}$ be a differentiable function on $[k,g]$ such that $\mathcal{L}' \in L^1[k,g]$, then the following equality holds*

$$ \begin{aligned} & \frac{1}{2+2\theta}\left(\mathcal{L}(k) + \theta\mathcal{L}\left(\frac{2k+g}{3}\right) + \theta\mathcal{L}\left(\frac{k+2g}{3}\right) + \mathcal{L}(g)\right) - \frac{3^{\alpha-1}\Gamma(\alpha+1)}{(g-k)^{\alpha}}\mathcal{R}(\alpha,\mathcal{L}) \\ = & \frac{g-k}{9}\left(\int_0^1 \left(\Lambda^{\alpha} - \frac{3}{2+2\theta}\right)\mathcal{L}'\left((1-\Lambda)k + \Lambda\frac{2k+g}{3}\right)d\Lambda \right. \\ & + \int_0^1 \left(\frac{1}{2} - (1-\Lambda)^{\alpha}\right)\mathcal{L}'\left((1-\Lambda)\frac{2k+g}{3} + \Lambda\frac{k+2g}{3}\right)d\Lambda \\ & \left. + \int_0^1 \left(\frac{3}{2+2\theta} - (1-\Lambda)^{\alpha}\right)\mathcal{L}'\left((1-\Lambda)\frac{k+2g}{3} + \Lambda g\right)d\Lambda\right), \end{aligned} $$

where θ is positive number, $0 < \alpha \leq 1$, and

$$ \mathcal{R}(\alpha,\mathcal{L}) = I^{\alpha}_{\left(\frac{2k+g}{3}\right)^-}\mathcal{L}(k) + I^{\alpha}_{\left(\frac{2k+g}{3}\right)^+}\mathcal{L}\left(\frac{k+2g}{3}\right) + I^{\alpha}_{\left(\frac{k+2g}{3}\right)^+}\mathcal{L}(g). \tag{2} $$

Proof. Let

$$ \begin{aligned} I_1 &= \int_0^1 \left(\Lambda^{\alpha} - \frac{3}{2+2\theta}\right)\mathcal{L}'\left((1-\Lambda)k + \Lambda\frac{2k+g}{3}\right)d\Lambda, \\ I_2 &= \int_0^1 \left(\frac{1}{2} - (1-\Lambda)^{\alpha}\right)\mathcal{L}'\left((1-\Lambda)\frac{2k+g}{3} + \Lambda\frac{k+2g}{3}\right)d\Lambda \end{aligned} $$

and

$$ I_3 = \int_0^1 \left(\frac{3}{2+2\theta} - (1-\Lambda)^{\alpha}\right)\mathcal{L}'\left((1-\Lambda)\frac{k+2g}{3} + \Lambda g\right)d\Lambda. $$

By using integration by parts in I_1, we obtain

$$ \begin{aligned} I_1 &= \frac{3}{g-k}\left(\Lambda^{\alpha} - \frac{3}{2+2\theta}\right)\mathcal{L}\left((1-\Lambda)k + \Lambda\frac{2k+g}{3}\right)\Big|_{\Lambda=0}^{\Lambda=1} \\ & \quad - \frac{3\alpha}{g-k}\int_0^1 \Lambda^{\alpha-1}\mathcal{L}\left((1-\Lambda)a + \Lambda\frac{2k+g}{3}\right)d\Lambda \\ &= \frac{6\theta-3}{(g-k)(2+2\theta)}\mathcal{L}\left(\frac{2k+g}{3}\right) + \frac{9}{(g-k)(2+2\theta)}\mathcal{L}(k) \\ & \quad - \frac{3^{\alpha+1}\alpha}{(g-k)^{\alpha+1}}\int_k^{\frac{2k+g}{3}} (u-a)^{\alpha-1}\mathcal{L}(u)du \\ &= \frac{6\theta-3}{(g-k)(2+2\theta)}\mathcal{L}\left(\frac{2k+g}{3}\right) + \frac{9}{(g-k)(2+2\theta)}\mathcal{L}(k) - \frac{3^{\alpha+1}\Gamma(\alpha+1)}{(g-k)^{\alpha+1}}I^{\alpha}_{\frac{2k+g}{3}^-}\mathcal{L}(k). \end{aligned} \tag{3} $$

Similarly, we obtain

$$
\begin{aligned}
I_2 &= \frac{3}{g-k}\left(\frac{1}{2}-(1-\Lambda)^{\alpha}\right)\mathcal{L}\left((1-\Lambda)\frac{2k+g}{3}+\Lambda\frac{k+2g}{3}\right)\Big|_{\Lambda=0}^{\Lambda=1} \\
&\quad -\frac{3\alpha}{g-k}\int_0^1(1-\Lambda)^{\alpha-1}\mathcal{L}\left((1-\Lambda)\frac{2k+g}{3}+\Lambda\frac{k+2g}{3}\right)d\Lambda \\
&= \frac{3}{2(g-k)}\mathcal{L}\left(\frac{k+2g}{3}\right)+\frac{3}{2(g-k)}\mathcal{L}\left(\frac{2k+g}{3}\right) \\
&\quad -\frac{3^{\alpha+1}\alpha}{(g-k)^{\alpha+1}}\int_{\frac{2k+g}{3}}^{\frac{k+2g}{3}}\left(\frac{k+2g}{3}-u\right)^{\alpha-1}\mathcal{L}(u)du \\
&= \frac{3}{2(g-k)}\mathcal{L}\left(\frac{k+2g}{3}\right)+\frac{3}{2(g-k)}\mathcal{L}\left(\frac{2k+g}{3}\right)-\frac{3^{\alpha+1}\Gamma(\alpha+1)}{(g-k)^{\alpha+1}}I^{\alpha}_{\left(\frac{2k+g}{3}\right)^{+}}\mathcal{L}\left(\frac{k+2g}{3}\right)
\end{aligned} \tag{4}
$$

and

$$
\begin{aligned}
I_3 &= \frac{3}{g-k}\left(\frac{3}{2+2\theta}-(1-\Lambda)^{\alpha}\right)\mathcal{L}\left((1-\Lambda)\frac{k+2g}{3}+\Lambda g\right)\Big|_{\Lambda=0}^{\Lambda=1} \\
&\quad -\frac{3\alpha}{g-k}\int_0^1(1-\Lambda)^{\alpha-1}\mathcal{L}\left((1-\Lambda)\frac{k+2g}{3}+\Lambda g\right)dt \\
&= \frac{9}{(2+2\theta)(g-k)}\mathcal{L}(g)+\frac{6\theta-3}{(g-k)(2+2\theta)}\mathcal{L}\left(\frac{k+2g}{3}\right) \\
&\quad -\frac{3^{\alpha+1}\alpha}{(g-k)^{\alpha+1}}\int_{\frac{k+2g}{3}}^{g}(g-u)^{\alpha-1}\mathcal{L}(u)du \\
&= \frac{9}{(g-k)(2+2\theta)}\mathcal{L}(g)+\frac{6\theta-3}{(g-k)(2+2\theta)}\mathcal{L}\left(\frac{k+2g}{3}\right)-\frac{3^{\alpha+1}\Gamma(\alpha+1)}{(g-k)^{\alpha+1}}I^{\alpha}_{\left(\frac{k+2g}{3}\right)^{+}}\mathcal{L}(g).
\end{aligned} \tag{5}
$$

Summing (3)–(5), and then multiplying the resulting equality by $\frac{g-k}{9}$, we obtain the desired result. □

Theorem 4. *Assume that all the assumptions of Lemma 1 are satisfied. Moreover, if $|\mathcal{L}'|$ is s-tgs-convex, then the following inequality holds*

$$
\begin{aligned}
&\left|\frac{1}{2+2\theta}\left(\mathcal{L}(k)+\theta\mathcal{L}\left(\frac{2k+g}{3}\right)+\theta\mathcal{L}\left(\frac{k+2g}{3}\right)+\mathcal{L}(g)\right)-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(g-k)^{\alpha}}\mathcal{R}(\alpha,\mathcal{L})\right| \\
\leq\ & \frac{g-k}{9}\Big(\varphi_\theta(s+1,s+1;\alpha)\left(|\mathcal{L}'(k)|+|\mathcal{L}'(g)|\right) \\
&+(\varphi_2(s+1,s+1;\alpha)+\varphi_\theta(s+1,s+1;\alpha))\left(\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|+\left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|\right)\Big),
\end{aligned}
$$

where $s\in[0,1]$, $\mathcal{R}(\alpha,\mathcal{L})$ is defined by (2),

$$\varkappa_\theta(x,y;\alpha)=B_{\left(\frac{3}{2+2\theta}\right)^{\frac{1}{\alpha}}}(x,y)-B_{1-\left(\frac{3}{2+2\theta}\right)^{\frac{1}{\alpha}}}(y,x) \tag{6}$$

and

$$\varphi_\theta(x,y;\alpha)=\frac{3}{2+2\theta}\varkappa_\theta(x,y;\alpha)-\varkappa_\theta(x+\alpha,y;\alpha). \tag{7}$$

Proof. By Lemma 1, modulus, and the s-tgs-convexity of $|\mathcal{L}'|$, we determine

$$\begin{aligned}
&\left|\frac{1}{2+2\theta}\left(\mathcal{L}(k)+\theta\mathcal{L}\left(\frac{2k+g}{3}\right)+\theta\mathcal{L}\left(\frac{k+2g}{3}\right)+\mathcal{L}(g)\right)-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(g-k)^{\alpha}}\mathcal{R}(\alpha,\mathcal{L})\right| \\
\leq\;& \frac{g-k}{9}\left(\int_0^1\left|\Lambda^{\alpha}-\frac{3}{2+2\theta}\right|\left|\mathcal{L}'\left((1-\Lambda)k+\Lambda\frac{2k+g}{3}\right)\right|d\Lambda\right. \\
&+\int_0^1\left|\frac{1}{2}-(1-\Lambda)^{\alpha}\right|\left|\mathcal{L}'\left((1-\Lambda)\frac{2k+g}{3}+\Lambda\frac{k+2g}{3}\right)\right|d\Lambda \\
&\left.+\int_0^1\left|\frac{3}{2+2\theta}-(1-\Lambda)^{\alpha}\right|\left|\mathcal{L}'\left((1-\Lambda)\frac{k+2g}{3}+\Lambda g\right)\right|d\Lambda\right) \\
\leq\;& \frac{g-k}{9}\left(\int_0^1\left|\Lambda^{\alpha}-\frac{3}{2+2\theta}\right|\Lambda^s(1-\Lambda)^s\left(|\mathcal{L}'(k)|+\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|\right)d\Lambda\right. \\
&+\int_0^1\left|\frac{1}{2}-(1-\Lambda)^{\alpha}\right|\Lambda^s(1-\Lambda)^s\left(\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|+\left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|\right)d\Lambda \\
&\left.+\int_0^1\left|\frac{3}{2+2\theta}-(1-\Lambda)^{\alpha}\right|\Lambda^s(1-\Lambda)^s\left(\left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|+|\mathcal{L}'(g)|\right)d\Lambda\right) \\
=\;& \frac{g-k}{9}\left(|\mathcal{L}'(k)|\int_0^1\left|\Lambda^{\alpha}-\frac{3}{2+2\theta}\right|\Lambda^s(1-\Lambda)^s dt\right. \\
&+\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right| \\
&\times\left(\int_0^1\left|\Lambda^{\alpha}-\frac{3}{2+2\theta}\right|\Lambda^s(1-\Lambda)^s d\Lambda+\int_0^1\left|\frac{1}{2}-(1-\Lambda)^{\alpha}\right|\Lambda^s(1-\Lambda)^s d\Lambda\right) \\
&+\left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right| \\
&\times\left(\int_0^1\left|\frac{1}{2}-(1-\Lambda)^{\alpha}\right|\Lambda^s(1-\Lambda)^s dt+\int_0^1\left|\frac{3}{2+2\theta}-(1-\Lambda)^{\alpha}\right|\Lambda^s(1-\Lambda)^s d\Lambda\right) \\
&\left.+|\mathcal{L}'(g)|\int_0^1\left|\frac{3}{2+2\theta}-(1-\Lambda)^{\alpha}\right|\Lambda^s(1-\Lambda)^s d\Lambda\right) \\
=\;& \frac{g-k}{9}\left(\varphi_{\theta}(s+1,s+1;\alpha)\left(|f'(k)|+|f'(g)|\right)\right. \\
&\left.+\left(\varphi_2(s+1,s+1;\alpha)+\varphi_{\theta}(s+1,s+1;\alpha)\right)\left(\left|f'\left(\frac{2k+g}{3}\right)\right|+\left|f'\left(\frac{k+2g}{3}\right)\right|\right)\right),
\end{aligned}$$

where $\varkappa_\theta$ and φ_θ are defined as in (6) and (7), respectively, and we have used the fact that

$$\begin{aligned}
&\int_0^1 \left|\Lambda^\alpha - \frac{3}{2+2\theta}\right|\Lambda^s(1-\Lambda)^s d\Lambda = \left|\frac{3}{2+2\theta} - (1-\Lambda)^\alpha\right|\Lambda^s(1-\Lambda)^s d\Lambda \\
= \;& \frac{3}{2+2\theta}\left(B_{\left(\frac{3}{2+2\theta}\right)^{\frac{1}{\alpha}}}(s+1,s+1) - B_{1-\left(\frac{3}{2+2\theta}\right)^{\frac{1}{\alpha}}}(s+1,s+1)\right) \\
& -\left(B_{\left(\frac{3}{2+2\theta}\right)^{\frac{1}{\alpha}}}(s+\alpha+1,s+1) - B_{1-\left(\frac{3}{2+2\theta}\right)^{\frac{1}{\alpha}}}(s+1,s+\alpha+1)\right) \\
= \;& \frac{3}{2+2\theta}\varkappa_\theta(s+1,s+1;\alpha) - \varkappa_\theta(s+\alpha+1,s+1;\alpha) \\
= \;& \varphi_\theta(s+1,s+1;\alpha)
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
\int_0^1 \left|\frac{1}{2} - (1-\Lambda)^\alpha\right|\Lambda^s(1-\Lambda)^s d\Lambda &= \int_0^1 \left|\Lambda^\alpha - \frac{1}{2}\right|\Lambda^s(1-\Lambda)^s d\Lambda \\
&= \varphi_2(s+1,s+1;\alpha),
\end{aligned} \tag{9}$$

which ends the proof. □

Corollary 1. *Taking $\alpha = 1$, Theorem 4 becomes*

$$\begin{aligned}
&\left|\frac{1}{2+2\theta}\left(\mathcal{L}(k) + \theta\mathcal{L}\left(\frac{2k+g}{3}\right) + \theta\mathcal{L}\left(\frac{k+2g}{3}\right) + \mathcal{L}(g)\right) - \frac{1}{g-k}\int_k^g \mathcal{L}(u)du\right| \\
\leq \;& \frac{g-k}{9}\left(\varphi_\theta(s+1,s+1;1)\left(|\mathcal{L}'(k)| + |\mathcal{L}'(g)|\right)\right. \\
& \left. + (\varphi_2(s+1,s+1;1) + \varphi_\theta(s+1,s+1;1))\left(\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right| + \left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|\right)\right),
\end{aligned}$$

where

$$\begin{aligned}
\varphi_\theta(s+1,s+1;1) &= \frac{3}{2+2\theta}B_{\frac{3}{2+2\theta}}(s+1,s+1) - B_{\frac{3}{2+2\theta}}(s+2,s+1) \\
&\quad + \frac{2\theta-1}{2+2\theta}B_{\frac{2\theta-1}{2+2\theta}}(s+1,s+1) - B_{\frac{2\theta-1}{2+2\theta}}(s+2,s+1)
\end{aligned}$$

and

$$\varphi_2(s+1,s+1;1) = B_{\frac{1}{2}}(s+1,s+1) - 2B_{\frac{1}{2}}(s+2,s+1).$$

Corollary 2. *Under the assumption of Theorem 4, and if $|f'|$ is P-function, then we have*

$$\begin{aligned}
&\left|\frac{1}{2+2\theta}\left(\mathcal{L}(k) + \theta\mathcal{L}\left(\frac{2k+g}{3}\right) + \theta\mathcal{L}\left(\frac{k+2g}{3}\right) + \mathcal{L}(g)\right) - \frac{3^{\alpha-1}\Gamma(\alpha+1)}{(g-k)^\alpha}\mathcal{R}(\alpha,\mathcal{L})\right| \\
\leq \;& \frac{g-k}{36}\left(\frac{(2\theta^2-2\theta+5)|\mathcal{L}'(k)| + (3\theta^2+6)\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right| + (3\theta^2+6)\left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right| + (2\theta^2-2\theta+5)|\mathcal{L}'(g)|}{(1+\theta)^2}\right).
\end{aligned}$$

Proof. Just replace s with 0 in Theorem 4. □

Remark 1. *In Corollary 2, choosing $\theta = 3$, we obtain the fractional Simpson's $3/8$ formula*

$$\begin{aligned}
&\left|\frac{1}{8}\left(\mathcal{L}(k) + 3\mathcal{L}\left(\frac{2k+g}{3}\right) + 3\mathcal{L}\left(\frac{k+2g}{3}\right) + \mathcal{L}(g)\right) - \frac{3^{\alpha-1}\Gamma(\alpha+1)}{(g-k)^\alpha}\mathcal{R}(\alpha,\mathcal{L})\right| \\
&\leq \frac{25(g-k)}{144}\left(\frac{17|\mathcal{L}'(k)| + 33\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right| + 33\left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right| + 17|\mathcal{L}'(g)|}{100}\right).
\end{aligned}$$

Remark 2. *In Corollary 2, choosing $\theta = \frac{27}{13}$, we obtain the fractional corrected Simpson's 3/8 formula*

$$\left|\frac{1}{80}\left(\mathcal{L}(k) + 27\mathcal{L}\left(\frac{2k+g}{3}\right) + 27\mathcal{L}\left(\frac{k+2g}{3}\right) + \mathcal{L}(g)\right) - \frac{3^{\alpha-1}\Gamma(\alpha+1)}{(g-k)^{\alpha}}\mathcal{R}(\alpha,\mathcal{L})\right| \leq \frac{2401(g-k)}{14400}\left(\frac{1601|\mathcal{L}'(k)|+3201\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|+3201\left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|+1601|\mathcal{L}'(g)|}{9604}\right).$$

Corollary 3. *In Corollary 2, if we take $\alpha = 1$, then we have*

$$\left|\frac{1}{2+2\theta}\left(\mathcal{L}(k) + \theta\mathcal{L}\left(\frac{2k+g}{3}\right) + \theta\mathcal{L}\left(\frac{k+2g}{3}\right) + \mathcal{L}(g)\right) - \frac{1}{g-k}\int_k^g \mathcal{L}(u)du\right| \leq \frac{g-k}{36}\left(\frac{(2\theta^2-2\theta+5)|\mathcal{L}'(k)|+(3\theta^2+6)\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|+(3\theta^2+6)\left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|+(2\theta^2-2\theta+5)|\mathcal{L}'(g)|}{(1+\theta)^2}\right).$$

Corollary 4. *Under the assumptions of Theorem 4, and if $|\mathcal{L}'|$ is tgs-function, then we have*

$$\begin{aligned}&\left|\frac{1}{2+2\theta}\left(\mathcal{L}(k) + \theta\mathcal{L}\left(\frac{2k+g}{3}\right) + \theta\mathcal{L}\left(\frac{k+2g}{3}\right) + \mathcal{L}(g)\right) - \frac{3^{\alpha-1}\Gamma(\alpha+1)}{(g-k)^{\alpha}}\mathcal{R}(\alpha,\mathcal{L})\right| \\ \leq\; &\frac{g-k}{1728}\left(\frac{27(1+4\theta)+(2\theta-1)^4}{(1+\theta)^4}\left(|\mathcal{L}'(k)| + |\mathcal{L}'(g)|\right)\right. \\ &\left.+\left(\frac{4(1+\theta)^4+27(1+4\theta)+(2\theta-1)^4}{(1+\theta)^4}\right)\left(\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right| + \left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|\right)\right).\end{aligned}$$

Proof. Just replace s with 1 in Theorem 4. □

Corollary 5. *Taking $\alpha = 1$ in Corollary 4 gives*

$$\begin{aligned}&\left|\frac{1}{2+2\theta}\left(\mathcal{L}(k) + \theta\mathcal{L}\left(\frac{2k+g}{3}\right) + \theta\mathcal{L}\left(\frac{k+2g}{3}\right) + \mathcal{L}(g)\right) - \frac{1}{g-k}\int_k^g \mathcal{L}(u)du\right| \\ \leq\; &\frac{g-k}{1728}\left(\frac{27(1+4\theta)+(2\theta-1)^4}{(1+\theta)^4}\left(|\mathcal{L}'(k)| + |\mathcal{L}'(g)|\right)\right. \\ &\left.+\left(\frac{4(1+\theta)^4+27(1+4\theta)+(2\theta-1)^4}{(1+\theta)^4}\right)\left(\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right| + \left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|\right)\right).\end{aligned}$$

Theorem 5. *Assume that all the assumptions of Theorem 4 are satisfied. Moreover, if $|\mathcal{L}'|^{\zeta}$ is s-tgs-convex, where $\zeta > 1$ with $\frac{1}{\tau} + \frac{1}{\zeta} = 1$, then the following inequality*

$$\begin{aligned}&\left|\frac{1}{2+2\theta}\left(\mathcal{L}(k) + \theta\mathcal{L}\left(\frac{2k+g}{3}\right) + \theta\mathcal{L}\left(\frac{k+2g}{3}\right) + \mathcal{L}(g)\right) - \frac{3^{\alpha-1}\Gamma(\alpha+1)}{(g-k)^{\alpha}}\mathcal{R}(\alpha,\mathcal{L})\right| \\ \leq\; &\frac{g-k}{9}(B(s+1,s+1))^{\frac{1}{\zeta}} \\ &\times\left(\left(\frac{B\left(\frac{1}{\alpha},\tau+1\right)}{\alpha}\left(\frac{3}{2+2\theta}\right)^{\tau+\frac{1}{\alpha}} + \frac{{}_2F_1\left(\frac{\alpha-1}{\alpha},1,\tau+2;\frac{2\theta-1}{2+2\theta}\right)}{\alpha(\tau+1)}\left(\frac{2\theta-1}{2+2\theta}\right)^{\tau+\frac{1}{\alpha}}\right)^{\frac{1}{\tau}}\right. \\ &\times\left(\left(|f'(k)|^{\zeta} + \left|f'\left(\frac{2k+g}{3}\right)\right|^{\zeta}\right)^{\frac{1}{\zeta}} + \left(\left|f'\left(\frac{k+2g}{3}\right)\right|^{\zeta} + |f'(g)|^{\zeta}\right)^{\frac{1}{\zeta}}\right) \\ &\left.+\left(\frac{B\left(\frac{1}{\alpha},\tau+1\right)}{2^{\tau+\frac{1}{\alpha}}\alpha} + \frac{{}_2F_1\left(\frac{\alpha-1}{\alpha},1,\tau+2;\frac{1}{2}\right)}{2^{\tau+1}\alpha(\tau+1)}\right)^{\frac{1}{\tau}}\left(\left|f'\left(\frac{2k+g}{3}\right)\right|^{\zeta} + \left|f'\left(\frac{k+2g}{3}\right)\right|^{\zeta}\right)^{\frac{1}{\zeta}}\right)\end{aligned}$$

holds, where $\mathcal{R}(\alpha,\mathcal{L})$ is defined by (2) B and ${}_2F_1$ are beta and hypergeometric functions, respectively.

Proof. By Lemma 1, modulus, Hölder's inequality, and the s-tgs-convexity of $|\mathcal{L}'|^{\zeta}$, we obtain

$$\begin{aligned}
&\left|\frac{1}{2+2\theta}\left(\mathcal{L}(k)+\theta\mathcal{L}\left(\frac{2k+g}{3}\right)+\theta\mathcal{L}\left(\frac{k+2g}{3}\right)+\mathcal{L}(g)\right)-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(g-k)^{\alpha}}\mathcal{R}(\alpha,\mathcal{L})\right| \\
\leq\ & \frac{g-k}{9}\left(\left(\int_0^1\left|\Lambda^{\alpha}-\frac{3}{2+2\theta}\right|^{\tau}d\Lambda\right)^{\frac{1}{\tau}}\left(\int_0^1\left|\mathcal{L}'\left((1-\Lambda)k+\Lambda\frac{2k+g}{3}\right)\right|^{\zeta}d\Lambda\right)^{\frac{1}{\zeta}}\right. \\
&+\left(\int_0^1\left|\frac{1}{2}-(1-\Lambda)^{\alpha}\right|^{\tau}d\Lambda\right)^{\frac{1}{\tau}}\left(\left(\int_0^1\left|\mathcal{L}'\left((1-\Lambda)\frac{2k+g}{3}+\Lambda\frac{k+2g}{3}\right)\right|^{\zeta}d\Lambda\right)^{\frac{1}{\zeta}}\right) \\
&\left.+\left(\int_0^1\left|\frac{3}{2+2\theta}-(1-\Lambda)^{\alpha}\right|^{\tau}d\Lambda\right)^{\frac{1}{\tau}}\left(\int_0^1\left|\mathcal{L}'\left((1-\Lambda)\frac{k+2g}{3}+\Lambda g\right)\right|^{\zeta}d\Lambda\right)^{\frac{1}{\zeta}}\right) \\
\leq\ & \frac{g-k}{9}\left(\left(\int_0^1\left|\Lambda^{\alpha}-\frac{3}{2+2\theta}\right|^{\tau}d\Lambda\right)^{\frac{1}{\tau}}\left(\int_0^1\Lambda^{s}(1-\Lambda)^{s}\left(\left|\mathcal{L}'(k)\right|^{\zeta}+\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|^{\zeta}\right)d\Lambda\right)^{\frac{1}{\zeta}}\right. \\
&+\left(\int_0^1\left|\frac{1}{2}-\Lambda^{\alpha}\right|^{\tau}d\Lambda\right)^{\frac{1}{\tau}}\left(\int_0^1\Lambda^{s}(1-\Lambda)^{s}\left(\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|^{\zeta}+\left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|^{\zeta}\right)d\Lambda\right)^{\frac{1}{\zeta}} \\
&\left.+\left(\int_0^1\left|\Lambda^{\alpha}-\frac{3}{2+2\theta}\right|^{\tau}d\Lambda\right)^{\frac{1}{\tau}}\left(\int_0^1\Lambda^{s}(1-\Lambda)^{s}\left(\left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|^{\zeta}+\left|\mathcal{L}'(g)\right|^{\zeta}\right)d\Lambda\right)^{\frac{1}{\zeta}}\right) \\
=\ & \frac{g-k}{9}(B(s+1,s+1))^{\frac{1}{\zeta}} \\
&\times\left(\left(\frac{1}{\alpha}\left(\frac{3}{2+2\theta}\right)^{\tau+\frac{1}{\alpha}}B\left(\frac{1}{\alpha},\tau+1\right)+\frac{{}_2F_1\left(\frac{\alpha-1}{\alpha},1,\tau+2;\frac{2\theta-1}{2+2\theta}\right)}{\alpha(\tau+1)}\left(\frac{2\theta-1}{2+2\theta}\right)^{\tau+\frac{1}{\alpha}}\right)^{\frac{1}{\tau}}\right. \\
&\times\left(\left(\left|\mathcal{L}'(k)\right|^{\zeta}+\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|^{\zeta}\right)^{\frac{1}{\zeta}}+\left(\left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|^{\zeta}+\left|\mathcal{L}'(g)\right|^{\zeta}\right)^{\frac{1}{\zeta}}\right) \\
&\left.+\left(\frac{B\left(\frac{1}{\alpha},\tau+1\right)}{2^{\tau+\frac{1}{\alpha}}\alpha}+\frac{{}_2F_1\left(\frac{\alpha-1}{\alpha},1,\tau+2;\frac{1}{2}\right)}{2^{\tau+1}\alpha(\tau+1)}\right)^{\frac{1}{\tau}}\left(\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|^{\zeta}+\left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|^{\zeta}\right)^{\frac{1}{\zeta}}\right),
\end{aligned}$$

where we used

$$\begin{aligned}
&\int_0^1\left|\Lambda^{\alpha}-\frac{3}{2+2\theta}\right|^{\tau}d\Lambda \\
=\ & \int_0^{\left(\frac{3}{2+2\theta}\right)^{\frac{1}{\alpha}}}\left(\frac{3}{2+2\theta}-\Lambda^{\alpha}\right)^{\tau}d\Lambda+\int_{\left(\frac{3}{2+2\theta}\right)^{\frac{1}{\alpha}}}^{1}\left(\Lambda^{\alpha}-\frac{3}{2+2\theta}\right)^{\tau}d\Lambda \\
=\ & \frac{1}{\alpha}\left(\frac{3}{2+2\theta}\right)^{\tau+\frac{1}{\alpha}}\int_0^1\vartheta^{\frac{1}{\alpha}-1}(1-\vartheta)^{\tau}d\vartheta+\frac{1}{\alpha}\left(\frac{2\theta-1}{2+2\theta}\right)^{\tau+\frac{1}{\alpha}}\int_0^1(1-\vartheta)^{\tau}\left(1-\frac{2\theta-1}{2+2\theta}\vartheta\right)^{\frac{1}{\alpha}-1}d\vartheta \\
=\ & \frac{1}{\alpha}\left(\frac{3}{2+2\theta}\right)^{\tau+\frac{1}{\alpha}}B\left(\frac{1}{\alpha},\tau+1\right)+\frac{1}{\alpha(\tau+1)}\left(\frac{2\theta-1}{2+2\theta}\right)^{\tau+\frac{1}{\alpha}}\cdot{}_2F_1\left(\frac{\alpha-1}{\alpha},1,\tau+2;\frac{2\theta-1}{2+2\theta}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \left|\frac{1}{2}-(1-\Lambda)^{\alpha}\right|^{\tau} d\Lambda \\
= &\int_0^{1-\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}} \left((1-\Lambda)^{\alpha}-\frac{1}{2}\right)^{\tau} d\Lambda + \int_{1-\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}^{1} \left(\frac{1}{2}-(1-\Lambda)^{\alpha}\right)^{\tau} d\Lambda \\
= &\frac{1}{2^{\tau+1}\alpha}\int_0^1 (1-x)^{\tau}\left(1-\frac{1}{2}x\right)^{\frac{1}{\alpha}-1} dx + \frac{1}{2^{\tau+\frac{1}{\alpha}}\alpha}\int_0^1 x^{\frac{1}{\alpha}-1}(1-x)^{\tau} dx. \\
= &\frac{1}{2^{\tau+1}\alpha(\tau+1)}\cdot {}_2F_1\left(\frac{\alpha-1}{\alpha},1,\tau+2;\frac{1}{2}\right) + \frac{1}{2^{\tau+\frac{1}{\alpha}}\alpha} B\left(\frac{1}{\alpha},\tau+1\right),
\end{aligned}$$

which ends the proof. □

Corollary 6. *Taking $\alpha = 1$ in Theorem 5, it yields*

$$\begin{aligned}
&\left|\frac{1}{2+2\theta}\left(\mathcal{L}(k)+\theta\mathcal{L}\left(\frac{2k+g}{3}\right)+\theta\mathcal{L}\left(\frac{k+2g}{3}\right)+\mathcal{L}(g)\right)-\frac{1}{g-k}\int_k^g \mathcal{L}(u)du\right| \\
\leq\ &\frac{g-k}{18}(B(s+1,s+1))^{\frac{1}{\zeta}}\left(\frac{1}{\tau+1}\right)^{\frac{1}{\tau}}\left(\left(\frac{3^{\tau+1}+(2\theta-1)^{\tau+1}}{2(1+\theta)^{\tau+1}}\right)^{\frac{1}{\tau}}\right. \\
&\times\left(\left(\left|\mathcal{L}'(k)\right|^{\zeta}+\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|^{\zeta}\right)^{\frac{1}{\zeta}}+\left(\left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|^{\zeta}+\left|\mathcal{L}'(g)\right|^{\zeta}\right)^{\frac{1}{\zeta}}\right) \\
&\left.+\left(\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|^{\zeta}+\left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|^{\zeta}\right)^{\frac{1}{\zeta}}\right).
\end{aligned}$$

Corollary 7. *Under the assumptions of Theorem 5, and if $|\mathcal{L}'|^{\zeta}$ is P-function, then we have*

$$\begin{aligned}
&\left|\frac{1}{2+2\theta}\left(\mathcal{L}(k)+\theta\mathcal{L}\left(\frac{2k+g}{3}\right)+\theta\mathcal{L}\left(\frac{k+2g}{3}\right)+\mathcal{L}(g)\right)-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(g-k)^{\alpha}}\mathcal{R}(\alpha,\mathcal{L})\right| \\
\leq\ &\frac{g-k}{9}\left(\left(\frac{B\left(\frac{1}{\alpha},\tau+1\right)}{\alpha}\left(\frac{3}{2+2\theta}\right)^{\tau+\frac{1}{\alpha}}+\frac{{}_2F_1\left(\frac{\alpha-1}{\alpha},1,\tau+2;\frac{2\theta-1}{2+2\theta}\right)}{\alpha(\tau+1)}\left(\frac{2\theta-1}{2+2\theta}\right)^{\tau+\frac{1}{\alpha}}\right)^{\frac{1}{\tau}}\right. \\
&\times\left(\left(\left|f'(k)\right|^{\zeta}+\left|f'\left(\frac{2k+g}{3}\right)\right|^{\zeta}\right)^{\frac{1}{\zeta}}+\left(\left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|^{\zeta}+\left|\mathcal{L}'(g)\right|^{\zeta}\right)^{\frac{1}{\zeta}}\right) \\
&\left.+\left(\frac{B\left(\frac{1}{\alpha},\tau+1\right)}{2^{\tau+\frac{1}{\alpha}}\alpha}+\frac{{}_2F_1\left(\frac{\alpha-1}{\alpha},1,\tau+2;\frac{1}{2}\right)}{2^{\tau+1}\alpha(\tau+1)}\right)^{\frac{1}{\tau}}\left(\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|^{\zeta}+\left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|^{\zeta}\right)^{\frac{1}{\zeta}}\right).
\end{aligned}$$

Proof. Just replace s with 0 in Theorem 5. □

Corollary 8. *In Corollary 7, if we take $\alpha = 1$, then we have*

$$
\begin{aligned}
&\left|\frac{1}{2+2\theta}\left(\mathcal{L}(k)+\theta\mathcal{L}\left(\frac{2k+g}{3}\right)+\theta\mathcal{L}\left(\frac{k+2g}{3}\right)+\mathcal{L}(g)\right)-\frac{1}{g-k}\int_k^g\mathcal{L}(u)du\right| \\
\leq\ & \frac{g-k}{18}\left(\frac{1}{\tau+1}\right)^{\frac{1}{\tau}}\left(\left(\frac{3^{\tau+1}+(2\theta-1)^{\tau+1}}{2(1+\theta)^{\tau+1}}\right)^{\frac{1}{\tau}}\right. \\
&\times\left(\left(\left|\mathcal{L}'(k)\right|^{\zeta}+\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|^{\zeta}\right)^{\frac{1}{\zeta}}+\left(\left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|^{\zeta}+\left|\mathcal{L}'(g)\right|^{\zeta}\right)^{\frac{1}{\zeta}}\right) \\
&\left.+\left(\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|^{\zeta}+\left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|^{\zeta}\right)^{\frac{1}{\zeta}}\right).
\end{aligned}
$$

Corollary 9. *Under the assumptions of Theorem 5, and if $|\mathcal{L}'|^{\zeta}$ is tgs-function, then we have*

$$
\begin{aligned}
&\left|\frac{1}{2+2\theta}\left(\mathcal{L}(k)+\theta\mathcal{L}\left(\frac{2k+g}{3}\right)+\theta\mathcal{L}\left(\frac{k+2g}{3}\right)+\mathcal{L}(g)\right)-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(g-k)^{\alpha}}\mathcal{R}(\alpha,\mathcal{L})\right| \\
\leq\ & \frac{g-k}{9}\left(\frac{1}{6}\right)^{\frac{1}{\zeta}}\left(\left(\frac{B\left(\frac{1}{\alpha},\tau+1\right)}{\alpha}\left(\frac{3}{2+2\theta}\right)^{\tau+\frac{1}{\alpha}}+\frac{{}_2F_1\left(\frac{\alpha-1}{\alpha},1,\tau+2;\frac{2\theta-1}{2+2\theta}\right)}{\alpha(\tau+1)}\left(\frac{2\theta-1}{2+2\theta}\right)^{\tau+\frac{1}{\alpha}}\right)^{\frac{1}{\tau}}\right. \\
&\times\left(\left(\left|\mathcal{L}'(k)\right|^{\zeta}+\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|^{\zeta}\right)^{\frac{1}{\zeta}}+\left(\left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|^{\zeta}+\left|\mathcal{L}'(g)\right|^{\zeta}\right)^{\frac{1}{\zeta}}\right) \\
&\left.+\left(\frac{B\left(\frac{1}{\alpha},\tau+1\right)}{2^{\tau+\frac{1}{\alpha}}\alpha}+\frac{{}_2F_1\left(\frac{\alpha-1}{\alpha},1,\tau+2;\frac{1}{2}\right)}{2^{\tau+1}\alpha(\tau+1)}\right)^{\frac{1}{\tau}}\left(\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|^{\zeta}+\left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|^{\zeta}\right)^{\frac{1}{\zeta}}\right).
\end{aligned}
$$

Proof. Just replace s with 1 in Theorem 5. □

Corollary 10. *In Corollary 9, if we take $\alpha = 1$, then we have*

$$
\begin{aligned}
&\left|\frac{1}{2+2\theta}\left(\mathcal{L}(k)+\theta\mathcal{L}\left(\frac{2k+g}{3}\right)+\theta\mathcal{L}\left(\frac{k+2g}{3}\right)+\mathcal{L}(g)\right)-\frac{1}{g-k}\int_k^g\mathcal{L}(u)du\right| \\
\leq\ & \frac{g-k}{18}\left(\frac{1}{6}\right)^{\frac{1}{\zeta}}\left(\frac{1}{\tau+1}\right)^{\frac{1}{\tau}}\left(\left(\frac{3^{\tau+1}+(2\theta-1)^{\tau+1}}{2(1+\theta)^{\tau+1}}\right)^{\frac{1}{\tau}}\right. \\
&\times\left(\left(\left|\mathcal{L}'(k)\right|^{\zeta}+\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|^{\zeta}\right)^{\frac{1}{\zeta}}+\left(\left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|^{\zeta}+\left|\mathcal{L}'(g)\right|^{\zeta}\right)^{\frac{1}{\zeta}}\right) \\
&\left.+\left(\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|^{\zeta}+\left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|^{\zeta}\right)^{\frac{1}{\zeta}}\right).
\end{aligned}
$$

Remark 3. *In Corollary 10, choosing $\theta = 3$, we obtain Simpson's $3/8$ formula*

$$\begin{aligned}
&\left|\frac{1}{8}\left(\mathcal{L}(k)+3\mathcal{L}\left(\frac{2k+g}{3}\right)+3\mathcal{L}\left(\frac{k+2g}{3}\right)+\mathcal{L}(g)\right)-\frac{1}{g-k}\int_k^g\mathcal{L}(u)du\right| \\
&\leq \frac{g-k}{72}\left(\frac{1}{6}\right)^{\frac{1}{\zeta}}\left(\frac{1}{\tau+1}\right)^{\frac{1}{\tau}}\left(\left(\frac{3^{\tau+1}+5^{\tau+1}}{8}\right)^{\frac{1}{\tau}}\right. \\
&\times\left(\left(\left|\mathcal{L}'(k)\right|^{\zeta}+\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|^{\zeta}\right)^{\frac{1}{\zeta}}+\left(\left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|^{\zeta}+\left|\mathcal{L}'(g)\right|^{\zeta}\right)^{\frac{1}{\zeta}}\right) \\
&\left.+\left(\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|^{\zeta}+\left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|^{\zeta}\right)^{\frac{1}{\zeta}}\right).
\end{aligned}$$

Remark 4. *In Corollary 10, choosing $\theta = \frac{27}{13}$, we obtain the corrected Simpson's $3/8$ formula*

$$\begin{aligned}
&\left|\frac{1}{80}\left(\mathcal{L}(k)+27\mathcal{L}\left(\frac{2k+g}{3}\right)+27\mathcal{L}\left(\frac{k+2g}{3}\right)+\mathcal{L}(g)\right)-\frac{1}{g-k}\int_k^g\mathcal{L}(u)du\right| \\
&\leq \frac{g-k}{18}\left(\frac{1}{12}\right)^{\frac{1}{\zeta}}\left(\frac{1}{\tau+1}\right)^{\frac{1}{\tau}}\left(\left(\frac{(39)^{\tau+1}+(41)^{\tau+1}}{(40)^{\tau+1}}\right)^{\frac{1}{\tau}}\right. \\
&\times\left(\left(\left|\mathcal{L}'(k)\right|^{\zeta}+\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|^{\zeta}\right)^{\frac{1}{\zeta}}+\left(\left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|^{\zeta}+\left|\mathcal{L}'(g)\right|^{\zeta}\right)^{\frac{1}{\zeta}}\right) \\
&\left.+\left(\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|^{\zeta}+\left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|^{\zeta}\right)^{\frac{1}{\zeta}}\right).
\end{aligned}$$

Theorem 6. *Assume that all the assumptions of Theorem 5 are satisfied. Moreover, if $|\mathcal{L}'|^{\zeta}$ is s-tgs-convex, where $\zeta \geq 1$, then the following inequality*

$$\begin{aligned}
&\left|\frac{1}{2+2\theta}\left(\mathcal{L}(k)+\theta\mathcal{L}\left(\frac{2k+g}{3}\right)+\theta\mathcal{L}\left(\frac{k+2g}{3}\right)+\mathcal{L}(g)\right)-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(g-k)^{\alpha}}\mathcal{R}(\alpha,\mathcal{L})\right| \\
\leq\;& \frac{g-k}{9}\left(\left(\frac{2\theta-3\alpha-1}{(2+2\theta)(\alpha+1)}+\frac{2\alpha}{\alpha+1}\left(\frac{3}{2+2\theta}\right)^{1+\frac{1}{\alpha}}\right)^{1-\frac{1}{\zeta}}(\varphi_{\theta}(s+1,s+1;\alpha))^{\frac{1}{\zeta}}\right. \\
&\times\left(\left(\left|\mathcal{L}'(k)\right|^{\zeta}+\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|^{\zeta}\right)^{\frac{1}{\zeta}}+\left(\left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|^{\zeta}+\left|\mathcal{L}'(g)\right|^{\zeta}\right)^{\frac{1}{\zeta}}\right) \\
&\left.+\left(\frac{1-\alpha+\alpha 2^{1-\frac{1}{\alpha}}}{2(\alpha+1)}\right)^{1-\frac{1}{\zeta}}(\varphi_2(s+1,s+1;\alpha))^{\frac{1}{\zeta}}\left(\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|^{\zeta}+\left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|^{\zeta}\right)^{\frac{1}{\zeta}}\right),
\end{aligned}$$

holds, where $\mathcal{R}(\alpha,\mathcal{L})$ and φ_θ are defined by (2) and (7), respectively.

Proof. By Lemma 1, modulus, power mean inequality, and the s-tgs-convexity of $|\mathcal{L}'|^\zeta$, we obtain

$$
\begin{aligned}
&\left|\frac{1}{2+2\theta}\left(\mathcal{L}(k)+\theta\mathcal{L}\left(\frac{2k+g}{3}\right)+\theta\mathcal{L}\left(\frac{k+2g}{3}\right)+\mathcal{L}(g)\right)-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(g-k)^\alpha}\mathcal{R}(\alpha,\mathcal{L})\right|\\
\leq\;& \frac{g-k}{9}\left(\left(\int_0^1\left|\Lambda^\alpha-\frac{3}{2+2\theta}\right|d\Lambda\right)^{1-\frac{1}{\zeta}}\left(\int_0^1\left|\Lambda^\alpha-\frac{3}{2+2\theta}\right|\left|\mathcal{L}'\left((1-\Lambda)k+\Lambda\frac{2k+g}{3}\right)\right|^\zeta d\Lambda\right)^{\frac{1}{\zeta}}\right.\\
&+\left(\int_0^1\left|\frac{1}{2}-(1-\Lambda)^\alpha\right|d\Lambda\right)^{1-\frac{1}{\zeta}}\\
&\times\left(\int_0^1\left|\frac{1}{2}-(1-\Lambda)^\alpha\right|\left|\mathcal{L}'\left((1-\Lambda)\frac{2k+g}{3}+\Lambda\frac{k+2g}{3}\right)\right|^\zeta d\Lambda\right)^{\frac{1}{\zeta}}\\
&+\left(\int_0^1\left|\frac{3}{2+2\theta}-(1-\Lambda)^\alpha\right|d\Lambda\right)^{1-\frac{1}{\zeta}}\\
&\left.\times\left(\int_0^1\left|\frac{3}{2+2\theta}-(1-\Lambda)^\alpha\right|\left|\mathcal{L}'\left((1-\Lambda)\frac{k+2g}{3}+\Lambda g\right)\right|^\zeta d\Lambda\right)^{\frac{1}{\zeta}}\right)\\
\leq\;& \frac{g-k}{9}\left(\left(\int_0^1\left|\Lambda^\alpha-\frac{3}{2+2\theta}\right|d\Lambda\right)^{1-\frac{1}{\zeta}}\left(\left|\mathcal{L}'(k)\right|^\zeta\int_0^1\left|\Lambda^\alpha-\frac{3}{2+2\theta}\right|\Lambda^s(1-\Lambda)^s d\Lambda\right.\right.\\
&\left.+\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|^\zeta\int_0^1\left|\Lambda^\alpha-\frac{3}{2+2\theta}\right|\Lambda^s(1-\Lambda)^s d\Lambda\right)^{\frac{1}{\zeta}}\\
&+\left(\int_0^1\left|\frac{1}{2}-\Lambda^\alpha\right|d\Lambda\right)^{1-\frac{1}{\zeta}}\left(\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|^\zeta\int_0^1\left|\frac{1}{2}-(1-\Lambda)^\alpha\right|\Lambda^s(1-\Lambda)^s d\Lambda\right.\\
&\left.+\left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|^\zeta\int_0^1\left|\frac{1}{2}-(1-\Lambda)^\alpha\right|\Lambda^s(1-\Lambda)^s d\Lambda\right)^{\frac{1}{\zeta}}\\
&+\left(\int_0^1\left|\frac{3}{2+2\theta}-\Lambda^\alpha\right|d\Lambda\right)^{1-\frac{1}{\zeta}}\left(\left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|^\zeta\int_0^1\left|\frac{3}{2+2\theta}-(1-\Lambda)^\alpha\right|\Lambda^s(1-\Lambda)^s d\Lambda\right.\\
&\left.\left.+\left|\mathcal{L}'(g)\right|^\zeta\int_0^1\left|\frac{3}{2+2\theta}-(1-\Lambda)^\alpha\right|\Lambda^s(1-\Lambda)^s d\Lambda\right)^{\frac{1}{\zeta}}\right)\\
=\;& \frac{g-k}{9}\left(\left(\frac{2\theta-3\alpha-1}{(2+2\theta)(\alpha+1)}+\frac{2\alpha}{\alpha+1}\left(\frac{3}{2+2\theta}\right)^{1+\frac{1}{\alpha}}\right)^{1-\frac{1}{\zeta}}(\varphi_\theta(s+1,s+1;\alpha))^{\frac{1}{\zeta}}\right.\\
&\times\left(\left(\left|\mathcal{L}'(k)\right|^\zeta+\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|^\zeta\right)^{\frac{1}{\zeta}}+\left(\left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|^\zeta+\left|\mathcal{L}'(g)\right|^\zeta\right)^{\frac{1}{\zeta}}\right)\\
&+\left(\frac{1-\alpha+\alpha 2^{1-\frac{1}{\alpha}}}{2(\alpha+1)}\right)^{1-\frac{1}{\zeta}}(\varphi_2(s+1,s+1;\alpha))^{\frac{1}{\zeta}}\left(\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|^\zeta+\left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|^\zeta\right)^{\frac{1}{\zeta}}\right),
\end{aligned}
$$

where we have considered (8) and (9). The proof is achieved. □

Corollary 11. *In Theorem 6, if we take $\alpha = 1$, then we have*

$$\begin{aligned} &\left| \frac{1}{2+2\theta}\left(\mathcal{L}(k) + \theta\mathcal{L}\left(\frac{2k+g}{3}\right) + \theta\mathcal{L}\left(\frac{k+2g}{3}\right) + \mathcal{L}(g)\right) - \frac{1}{g-k}\int_k^g \mathcal{L}(u)du \right| \\ \leq\ & \frac{g-k}{9}\left(\left(\frac{2\theta^2-2\theta+5}{(2+2\theta)^2}\right)^{1-\frac{1}{\zeta}}(\varphi_\theta(s+1,s+1;1))^{\frac{1}{\zeta}} \right. \\ &\times\left(\left(\left|\mathcal{L}'(k)\right|^\zeta + \left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|^\zeta\right)^{\frac{1}{\zeta}} + \left(\left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|^\zeta + \left|\mathcal{L}'(g)\right|^\zeta\right)^{\frac{1}{\zeta}}\right) \\ &\left. + \left(\frac{1}{4}\right)^{1-\frac{1}{\zeta}}(\varphi_2(s+1,s+1;1))^{\frac{1}{\zeta}}\left(\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|^\zeta + \left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|^\zeta\right)^{\frac{1}{\zeta}}\right). \end{aligned}$$

Corollary 12. *Under the assumptions of Theorem 6, and if $|\mathcal{L}'|^\zeta$ is P-function, then we have*

$$\begin{aligned} &\left| \frac{1}{2+2\theta}\left(\mathcal{L}(k) + \theta\mathcal{L}\left(\frac{2k+g}{3}\right) + \theta\mathcal{L}\left(\frac{k+2g}{3}\right) + \mathcal{L}(g)\right) - \frac{3^{\alpha-1}\Gamma(\alpha+1)}{(g-k)^\alpha}\mathcal{R}(\alpha,\mathcal{L}) \right| \\ \leq\ & \frac{g-k}{9}\left(\left(\frac{2\theta-3\alpha-1}{(2+2\theta)(\alpha+1)} + \frac{2\alpha}{\alpha+1}\left(\frac{3}{2+2\theta}\right)^{1+\frac{1}{\alpha}}\right) \right. \\ &\times\left(\left(\left|\mathcal{L}'(k)\right|^\zeta + \left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|^\zeta\right)^{\frac{1}{\zeta}} + \left(\left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|^\zeta + \left|\mathcal{L}'(g)\right|^\zeta\right)^{\frac{1}{\zeta}}\right) \\ &\left. + \frac{1-\alpha+\alpha 2^{1-\frac{1}{\alpha}}}{2(\alpha+1)}\left(\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|^\zeta + \left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|^\zeta\right)^{\frac{1}{\zeta}}\right). \end{aligned}$$

Proof. Just replace s with 0 in Theorem 6. □

Corollary 13. *In Corollary 12, if we take $\alpha = 1$, then we have*

$$\begin{aligned} &\left| \frac{1}{2+2\theta}\left(\mathcal{L}(k) + \theta\mathcal{L}\left(\frac{2k+g}{3}\right) + \theta\mathcal{L}\left(\frac{k+2g}{3}\right) + \mathcal{L}(g)\right) - \frac{1}{g-k}\int_k^g \mathcal{L}(u)du \right| \\ \leq\ & \frac{g-k}{36}\left(\frac{2\theta^2-2\theta+5}{(1+\theta)^2}\left(\left(\left|\mathcal{L}'(k)\right|^\zeta + \left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|^\zeta\right)^{\frac{1}{\zeta}} + \left(\left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|^\zeta + \left|\mathcal{L}'(g)\right|^\zeta\right)^{\frac{1}{\zeta}}\right) \right. \\ &\left. + \left(\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|^\zeta + \left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|^\zeta\right)^{\frac{1}{\zeta}}\right). \end{aligned}$$

Corollary 14. *Under the assumptions of Theorem 6, and if $|\mathcal{L}'|^{\zeta}$ is tgs-function, then we have*

$$\begin{aligned} &\left|\frac{1}{2+2\theta}\left(\mathcal{L}(k)+\theta\mathcal{L}\left(\frac{2k+g}{3}\right)+\theta\mathcal{L}\left(\frac{k+2g}{3}\right)+\mathcal{L}(g)\right)-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(g-k)^{\alpha}}\mathcal{R}(\alpha,\mathcal{L})\right| \\ \leq\ & \frac{g-k}{9}\left(\left(\frac{2\theta-3\alpha-1}{(2+2\theta)(\alpha+1)}+\frac{2\alpha}{\alpha+1}\left(\frac{3}{2+2\theta}\right)^{1+\frac{1}{\alpha}}\right)^{1-\frac{1}{\zeta}}\right. \\ &\times\left(\frac{4\theta-\alpha^2-5\alpha-2}{4(1+\theta)(\alpha+2)(\alpha+3)}+\frac{\alpha}{\alpha+2}\left(\frac{3}{2+2\theta}\right)^{1+\frac{2}{\alpha}}-\frac{2\alpha}{3(\alpha+3)}\left(\frac{3}{2+2\theta}\right)^{1+\frac{3}{\alpha}}\right)^{\frac{1}{\zeta}} \\ &\times\left(\left(\left|\mathcal{L}'(k)\right|^{\zeta}+\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|^{\zeta}\right)^{\frac{1}{\zeta}}+\left(\left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|^{\zeta}+\left|\mathcal{L}'(g)\right|^{\zeta}\right)^{\frac{1}{\zeta}}\right) \\ &+\left(\frac{1-\alpha+\alpha2^{1-\frac{1}{\alpha}}}{2(\alpha+1)}\right)^{1-\frac{1}{\zeta}}\left(\frac{6-5\alpha-\alpha^2}{12(\alpha+2)(\alpha+3)}+\frac{\alpha}{2(\alpha+2)}\left(\frac{1}{2}\right)^{\frac{2}{\alpha}}-\frac{\alpha}{3(\alpha+3)}\left(\frac{1}{2}\right)^{\frac{3}{\alpha}}\right)^{\frac{1}{\zeta}} \\ &\left.\times\left(\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|^{\zeta}+\left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|^{\zeta}\right)^{\frac{1}{\zeta}}\right). \end{aligned}$$

Proof. Just replace s with 1 in Theorem 6. □

Corollary 15. *In Corollary 14, if we take $\alpha = 1$, then we have*

$$\begin{aligned} &\left|\frac{1}{2+2\theta}\left(\mathcal{L}(k)+\theta\mathcal{L}\left(\frac{2k+g}{3}\right)+\theta\mathcal{L}\left(\frac{k+2g}{3}\right)+\mathcal{L}(g)\right)-\frac{1}{g-k}\int_k^g\mathcal{L}(u)du\right| \\ \leq\ & \frac{g-k}{36}\left(\frac{2\theta^2-2\theta+5}{(1+\theta)^2}\left(\frac{(\theta-2)(1+\theta)^2}{3(2\theta^2-2\theta+5)}+\frac{36(1+\theta)-27}{8(2\theta^2-2\theta+5)(1+\theta)^2}\right)^{\frac{1}{\zeta}}\right. \\ &\times\left(\left(\left|\mathcal{L}'(k)\right|^{\zeta}+\left|f'\left(\frac{2k+g}{3}\right)\right|^{\zeta}\right)^{\frac{1}{\zeta}}+\left(\left|f'\left(\frac{k+2g}{3}\right)\right|^{\zeta}+\left|\mathcal{L}'(g)\right|^{\zeta}\right)^{\frac{1}{\zeta}}\right) \\ &\left.+\left(\frac{1}{8}\right)^{\frac{1}{\zeta}}\left(\left|\mathcal{L}'\left(\frac{2k+g}{3}\right)\right|^{\zeta}+\left|\mathcal{L}'\left(\frac{k+2g}{3}\right)\right|^{\zeta}\right)^{\frac{1}{\zeta}}\right). \end{aligned}$$

3. Applications

Let Y be the division of points $k = \psi_0 < \psi_1 <\ldots< \psi_n = g$ of $[k, g]$, and consider the following formula for quadrature

$$\int_k^g \mathcal{L}(w)dw = \lambda(\mathcal{L}, \mathrm{Y}) + R(\mathcal{L}, \mathrm{Y}),$$

where

$$\lambda(\mathcal{L},\mathrm{Y})=\sum_{\epsilon=0}^{n-1}\frac{\psi_{\epsilon+1}-\psi_{\epsilon}}{2+2\theta}\left(\mathcal{L}(\psi_{\epsilon})+\theta\mathcal{L}\left(\frac{2\psi_{\epsilon}+\psi_{\epsilon+1}}{3}\right)+\theta\mathcal{L}\left(\frac{\psi_{\epsilon}+2\psi_{\epsilon+1}}{3}\right)+\mathcal{L}(\psi_{\epsilon+1})\right)$$

and $R(\mathcal{L}, \mathrm{Y})$ represents the associated approximation error.

Proposition 1. *Let $\mathcal{L}$ be as in Lemma 1 and $n \in \mathbb{N}$. If $|\mathcal{L}'|$ is P-function, we have*

$$|R(\mathcal{L},\mathrm{Y})| \le \sum_{\epsilon=0}^{n-1} \frac{(\psi_{\epsilon+1}-\psi_\epsilon)^2}{36}\left(\frac{2\theta^2-2\theta+5}{(1+\theta)^2}\left(|\mathcal{L}'(\psi_\epsilon)| + |\mathcal{L}'(\psi_{\epsilon+1})|\right)\right.$$
$$\left. + \frac{3\theta^2+6}{(1+\theta)^2}\left(\left|\mathcal{L}'\left(\frac{2\psi_\epsilon+\psi_{\epsilon+1}}{3}\right)\right| + \left|\mathcal{L}'\left(\frac{\psi_\epsilon+2\psi_{\epsilon+1}}{3}\right)\right|\right)\right).$$

Proof. Applying Corollary 3 on $[\psi_\epsilon, \psi_{\epsilon+1}]$ $(\epsilon = 0, 1, \ldots, n-1)$ of the partition Y, we determine

$$\left|\frac{\mathcal{L}(\psi_\epsilon)+\theta\mathcal{L}\left(\frac{2\psi_\epsilon+\psi_{\epsilon+1}}{3}\right)+\theta\mathcal{L}\left(\frac{\psi_\epsilon+2\psi_{\epsilon+1}}{3}\right)+\mathcal{L}(\psi_{\epsilon+1})}{2+2\theta} - \frac{1}{\psi_{\epsilon+1}-\psi_\epsilon}\int_{\psi_\epsilon}^{\psi_{\epsilon+1}} \mathcal{L}(w)dw\right|$$
$$\le \frac{\psi_{\epsilon+1}-\psi_\epsilon}{36}\left(\frac{2\theta^2-2\theta+5}{(1+\theta)^2}\left(|\mathcal{L}'(\psi_\epsilon)| + |\mathcal{L}'(\psi_{\epsilon+1})|\right)\right.$$
$$\left. + \frac{3\theta^2+6}{(1+\theta)^2}\left(\left|\mathcal{L}'\left(\frac{2\psi_\epsilon+\psi_{\epsilon+1}}{3}\right)\right| + \left|\mathcal{L}'\left(\frac{\psi_\epsilon+2\psi_{\epsilon+1}}{3}\right)\right|\right)\right).$$

The desired inequality follows by multiplying the above inequality by $(\psi_{\epsilon+1} - \psi_\epsilon)$, then adding the result over $\epsilon = 0, 1, \ldots, n-1$ and using the triangle inequality. □

Proposition 2. *Let $\mathcal{L}$ be as in Lemma 1 and $n \in \mathbb{N}$. If $|\mathcal{L}'|$ is tgs-convex, we have*

$$|R(\mathcal{L},\mathrm{Y})|$$
$$\le \sum_{i=0}^{n-1} \frac{(\psi_{\epsilon+1}-\psi_\epsilon)^2}{1728}\left(\frac{27(1+4\theta)+(2\theta-1)^4}{(1+\theta)^4}\left(|\mathcal{L}'(\psi_\epsilon)| + |\mathcal{L}'(\psi_{\epsilon+1})|\right)\right.$$
$$\left. + \left(\frac{4(1+\theta)^4+27(1+4\theta)+(2\theta-1)^4}{(1+\theta)^4}\right)\left(\left|\mathcal{L}'\left(\frac{2\psi_\epsilon+\psi_{\epsilon+1}}{3}\right)\right| + \left|\mathcal{L}'\left(\frac{\psi_\epsilon+2\psi_{\epsilon+1}}{3}\right)\right|\right)\right).$$

Proof. Applying Corollary 5 on $[\psi_\epsilon, \psi_{\epsilon+1}]$ $(\epsilon = 0, 1, \ldots, n-1)$ of the partition Y, we obtain

$$\left|\frac{\mathcal{L}(\psi_\epsilon)+\theta\mathcal{L}\left(\frac{2\psi_\epsilon+\psi_{\epsilon+1}}{3}\right)+\theta\mathcal{L}\left(\frac{\psi_\epsilon+2\psi_{\epsilon+1}}{3}\right)+\mathcal{L}(\psi_{\epsilon+1})}{2+2\theta} - \frac{1}{\psi_{\epsilon+1}-\psi_\epsilon}\int_{\psi_\epsilon}^{\psi_{\epsilon+1}} \mathcal{L}(w)dw\right|$$
$$\le \frac{\psi_{\epsilon+1}-\psi_\epsilon}{1728}\left(\frac{27(1+4\theta)+(2\theta-1)^4}{(1+\theta)^4}\left(|\mathcal{L}'(\psi_\epsilon)| + |\mathcal{L}'(\psi_{\epsilon+1})|\right)\right.$$
$$\left. + \left(\frac{4(1+\theta)^4+27(1+4\theta)+(2\theta-1)^4}{(1+\theta)^4}\right)\left(\left|\mathcal{L}'\left(\frac{2\psi_\epsilon+\psi_{\epsilon+1}}{3}\right)\right| + \left|\mathcal{L}'\left(\frac{\psi_\epsilon+2\psi_{\epsilon+1}}{3}\right)\right|\right)\right).$$

The desired inequality follows by multiplying the above inequality by $(\psi_{\epsilon+1} - \psi_\epsilon)$, then adding the result over ϵ =0, 1, . . ., $n-1$ and using the triangle inequality. □

Proposition 3. *Let $\mathcal{L}$ be as in Lemma 1 and $n \in \mathbb{N}$. If $|\mathcal{L}'|^\zeta$ is P-convex where $\zeta, \tau > 1$ with $\frac{1}{\tau} + \frac{1}{\zeta} = 1$, we have*

$$|R(\mathcal{L},\mathrm{Y})|$$
$$\le \sum_{i=0}^{n-1} \frac{(\psi_{\epsilon+1}-\psi_\epsilon)^2}{18}\left(\frac{1}{\tau+1}\right)^{\frac{1}{\tau}}\left(\left(\frac{3^{\tau+1}+(2\theta-1)^{\tau+1}}{2(1+\theta)^{\tau+1}}\right)^{\frac{1}{\tau}}\right.$$
$$\times\left(\left(|\mathcal{L}'(\psi_\epsilon)|^\zeta + \left|\mathcal{L}'\left(\frac{2\psi_\epsilon+\psi_{\epsilon+1}}{3}\right)\right|^\zeta\right)^{\frac{1}{\zeta}} + \left(\left|\mathcal{L}'\left(\frac{\psi_\epsilon+2\psi_{\epsilon+1}}{3}\right)\right|^\zeta + |\mathcal{L}'(\psi_{\epsilon+1})|^\zeta\right)^{\frac{1}{\zeta}}\right)$$
$$\left. + \left(\left|\mathcal{L}'\left(\frac{2\psi_\epsilon+\psi_{\epsilon+1}}{3}\right)\right|^\zeta + \left|\mathcal{L}'\left(\frac{\psi_\epsilon+2\psi_{\epsilon+1}}{3}\right)\right|^\zeta\right)^{\frac{1}{\zeta}}\right).$$

Proof. Applying Corollary 8 on $[\psi_\epsilon, \psi_{\epsilon+1}]$ $(\epsilon = 0, 1, \ldots, n-1)$ of the partition Y, we determine

$$\begin{aligned}
&\left|\frac{\mathcal{L}(\psi_\epsilon)+\theta\mathcal{L}\left(\frac{2\psi_\epsilon+\psi_{\epsilon+1}}{3}\right)+\theta\mathcal{L}\left(\frac{\psi_\epsilon+2\psi_{\epsilon+1}}{3}\right)+\mathcal{L}(\psi_{\epsilon+1})}{2+2\theta} - \frac{1}{\psi_{\epsilon+1}-\psi_\epsilon}\int_{\psi_\epsilon}^{\psi_{\epsilon+1}} \mathcal{L}(w)dw\right| \\
\leq\ & \frac{\psi_{\epsilon+1}-\psi_\epsilon}{18}\left(\frac{1}{\tau+1}\right)^{\frac{1}{\tau}}\left(\left(\frac{3^{\tau+1}+(2\theta-1)^{\tau+1}}{2(1+\theta)^{\tau+1}}\right)^{\frac{1}{\tau}}\right. \\
&\times\left(\left(\left|\mathcal{L}'(\psi_\epsilon)\right|^\zeta + \left|\mathcal{L}'\left(\frac{2\psi_\epsilon+\psi_{\epsilon+1}}{3}\right)\right|^\zeta\right)^{\frac{1}{\zeta}} + \left(\left|\mathcal{L}'\left(\frac{\psi_\epsilon+2\psi_{\epsilon+1}}{3}\right)\right|^\zeta + \left|\mathcal{L}'(\psi_{\epsilon+1})\right|^\zeta\right)^{\frac{1}{\zeta}}\right) \\
&\left.+\left(\left|\mathcal{L}'\left(\frac{2\psi_\epsilon+\psi_{\epsilon+1}}{3}\right)\right|^\zeta + \left|\mathcal{L}'\left(\frac{\psi_\epsilon+2\psi_{\epsilon+1}}{3}\right)\right|^\zeta\right)^{\frac{1}{\zeta}}\right).
\end{aligned}$$

The desired inequality follows by multiplying the above inequality by $(\psi_{\epsilon+1} - \psi_\epsilon)$, then adding the result over $\epsilon = 0, 1, \ldots, n-1$ and using the triangle inequality. □

Proposition 4. *Let $\mathcal{L}$ be as in Lemma 1 and $n \in \mathbb{N}$. If $|\mathcal{L}'|^\zeta$ is tgs-convex where $\zeta, \tau > 1$ with $\frac{1}{\tau} + \frac{1}{\zeta} = 1$, we have*

$$\begin{aligned}
&|R(f, Y)| \\
\leq\ & \sum_{i=0}^{n-1}\frac{(\psi_{\epsilon+1}-\psi_\epsilon)^2}{18}\left(\frac{1}{6}\right)^{\frac{1}{\zeta}}\left(\frac{1}{\tau+1}\right)^{\frac{1}{\tau}}\left(\left(\frac{3^{\tau+1}+(2\theta-1)^{\tau+1}}{2(1+\theta)^{\tau+1}}\right)^{\frac{1}{\tau}}\right. \\
&\times\left(\left(\left|\mathcal{L}'(\psi_\epsilon)\right|^\zeta + \left|\mathcal{L}'\left(\frac{2\psi_\epsilon+\psi_{\epsilon+1}}{3}\right)\right|^\zeta\right)^{\frac{1}{\zeta}} + \left(\left|\mathcal{L}'\left(\frac{\psi_\epsilon+2\psi_{\epsilon+1}}{3}\right)\right|^\zeta + \left|\mathcal{L}'(\psi_{\epsilon+1})\right|^\zeta\right)^{\frac{1}{\zeta}}\right) \\
&\left.+\left(\left|\mathcal{L}'\left(\frac{2\psi_\epsilon+\psi_{\epsilon+1}}{3}\right)\right|^\zeta + \left|\mathcal{L}'\left(\frac{\psi_\epsilon+2\psi_{\epsilon+1}}{3}\right)\right|^\zeta\right)^{\frac{1}{\zeta}}\right).
\end{aligned}$$

Proof. Applying Corollary 10 on $[\psi_\epsilon, \psi_{\epsilon+1}]$ $(\epsilon = 0, 1, \ldots, n-1)$ of the partition Y, we obtain

$$\begin{aligned}
&\left|\frac{\mathcal{L}(\psi_\epsilon)+\theta\mathcal{L}\left(\frac{2\psi_\epsilon+\psi_{\epsilon+1}}{3}\right)+\theta\mathcal{L}\left(\frac{\psi_\epsilon+2\psi_{\epsilon+1}}{3}\right)+\mathcal{L}(\psi_{\epsilon+1})}{2+2\theta} - \frac{1}{\psi_{\epsilon+1}-\psi_\epsilon}\int_{\psi_\epsilon}^{\psi_{\epsilon+1}} \mathcal{L}(w)dw\right| \\
\leq\ & \frac{\psi_{\epsilon+1}-\psi_\epsilon}{18}\left(\frac{1}{6}\right)^{\frac{1}{\zeta}}\left(\frac{1}{\tau+1}\right)^{\frac{1}{\tau}}\left(\left(\frac{3^{\tau+1}+(2\theta-1)^{\tau+1}}{2(1+\theta)^{\tau+1}}\right)^{\frac{1}{\tau}}\right. \\
&\times\left(\left(\left|\mathcal{L}'(\psi_\epsilon)\right|^\zeta + \left|\mathcal{L}'\left(\frac{2\psi_\epsilon+\psi_{\epsilon+1}}{3}\right)\right|^\zeta\right)^{\frac{1}{\zeta}} + \left(\left|\mathcal{L}'\left(\frac{\psi_\epsilon+2\psi_{\epsilon+1}}{3}\right)\right|^\zeta + \left|\mathcal{L}'(\psi_{\epsilon+1})\right|^\zeta\right)^{\frac{1}{\zeta}}\right) \\
&\left.+\left(\left|\mathcal{L}'\left(\frac{2\psi_\epsilon+\psi_{\epsilon+1}}{3}\right)\right|^\zeta + \left|\mathcal{L}'\left(\frac{\psi_\epsilon+2\psi_{\epsilon+1}}{3}\right)\right|^\zeta\right)^{\frac{1}{\zeta}}\right).
\end{aligned}$$

The desired inequality follows by multiplying the above inequality by $(\psi_{\epsilon+1} - \psi_\epsilon)$, then adding the result over $\epsilon = 0, 1, \ldots, n-1$ and using the triangle inequality. □

Let us consider the following means for arbitrary real numbers k, g
The Arithmetic mean: $A(k, g, n) = \frac{k+g+n}{3}$.
The Harmonic mean: $H(k, g, n) = \frac{3}{\frac{1}{k}+\frac{1}{g}+\frac{1}{n}}$
The Geometric means: $G(k, g) = \sqrt{kg}$
The p-Logarithmic mean: $L_p(k, g) = \left(\frac{g^{p+1}-k^{p+1}}{(p+1)(g-k)}\right)^{\frac{1}{p}}, k, g > 0, k \neq g,$ and $p \in \mathbb{R}\setminus\{-1, 0\}$.

Proposition 5. *Let $k, g \in \mathbb{R}$ with $0 < k < g$, then we have*

$$\left|\frac{1}{4}\left(2A\left(g^{-3}, k^{-3}\right) + H^{-1}(g,g,k) + H^{-1}(g,k,k)\right) - 4G^{-6}(k,g)L_3^3(k,g)\right|$$
$$\le \frac{g-k}{12kg}\left(\frac{5}{g^2} + \left(\frac{2k+g}{kg}\right)^2 + \left(\frac{k+2g}{kg}\right)^2 + \frac{5}{k^2}\right).$$

Proof. The assertion follows from Corollary 2 with $\alpha = 1$ and $\theta = 1$, applied to the function $\mathcal{L}(w) = w^3$ on $\left[\frac{1}{g}, \frac{1}{k}\right]$. □

Proposition 6. *Let $k, g \in \mathbb{R}$ with $0 < k < g$, then we have*

$$\left|2A\left(k^2, g^2\right) + 3A^2(k,k,g) + 3A^2(k,g,g) - 8L_2^2(k,g)\right|$$
$$\le \frac{\sqrt{57}(g-k)}{27}\left(\frac{\left(13k^2+4kg+g^2\right)^{\frac{1}{2}} + \left(k^2+4kg+13g^2\right)^{\frac{1}{2}}}{12} + \frac{\left(5k^2+8kg+5g^2\right)^{\frac{1}{2}}}{3\sqrt{19}}\right).$$

Proof. The assertion follows from Corollary 8 with $\theta = 3$ and $\zeta = 2$, applied to the function $\mathcal{L}(w) = \frac{1}{2}w^2$ on $[k, g]$, in which $|\mathcal{L}'(w)|^2 = w^2$ is P-function. □

4. Conclusions

In this study, we have considered the fractional Newton–Cotes type integral inequalities involving four points via a Riemann–Liouville integral operator. We have established for the first time a novel parametrized integral identity. Based on this equality, we have derived several 3/8 Simpson-like type inequalities for functions whose first derivatives belongs in the class of s-tgs-convex functions. Some special cases are discussed according to the values of parameters. Some applications to numerical quadratures are presented. The obtained results may lead to additional research in this fascinating field as well as generalizations in other types of calculations, including multiplicative calculus and quantum calculus.

Author Contributions: Conceptualization, M.M., B.M., H.B. and A.M.; Methodology, M.M., B.M., H.B. and A.M.; Writing—original draft, M.M., B.M., H.B. and A.M.; Writing—review and editing, M.M., B.M., H.B., A.M. and M.B.; Project administration, B.M., H.B., A.M. and M.B.; Funding acquisition, A.M. and M.B. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Deanship of Scientific Research at King Khalid University through large research project under grant number R.G.P.2/252/44.

Data Availability Statement: Not applicable.

Acknowledgments: The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through large research project under grant number R.G.P.2/252/44.

Conflicts of Interest: The authors declare no conflict of interest.

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