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Full Length Article

New algorithm to solve nonlinear functional equations applying linearization then double discretization scheme (L.D.D)

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ABSTRACT

Solving functional nonlinear equations leads to the question: which to begin with, linearization or discretization? Recent papers confirmed that linearizing then discretizing (L.D) is better. In this paper, we develop a new numerical scheme that begins with the linearization phase and then, a double discretizations process (L.D.D). This new method gives a different theoretical framework where the convergence process is satisfied under some hypotheses. Many examples are offered to show the effectiveness of our new scheme. Starting with a comparison of the obtained numerical results with the results of a recently published research, moreover, applications to the system of nonlinear integro-differential equations. Obtained numerical results show that our (L.D.D) method is more efficient for solving nonlinear functional equations.

Introduction

Integral and integro-differential equations appear naturally in many fields of applied mathematics, where multiple theoretical and numerical studies are well investigated. See, e.g. Atkinson et al. [1]; Nair [2]; Fernandez et al. [3]; Cakir et al. [4]. In the complex Banach space χ, let $F : \mathcal{V}\subset\chi \rightarrow \chi$ be a nonlinear Fréchet differentiable operator defined on a nonempty open set $\mathcal{V}$ of χ. In general, the nonlinear integro-differential equations are set as

$$\text{Find } \psi \in \mathcal{V} : \quad F(\psi) = 0_{\chi}. \tag{1}$$

Many methods have been constructed to find the approximate solution to these problems. See, Chakraborty et al. [5]; Mundewadi et al. [6]; Katani [7]. To solve this class of nonlinear functional equations, we generally use the classical process, which depends firstly on the discretization of equation (1), so we obtain a nonlinear algebraic system, and then the linearizations of these discrete nonlinear equations using, for example, Newton's iteration method or Banach's iteration method. For instance, many results concerning the classical process, denoted (D.L), have been achieved to solve these kinds of equations. See, e.g. Bounaya et al. [8]; Touati et al. [9]; Mirzaee et al. [10].

In a recent paper, Grammont et al. [11], the authors develop a numerical process called the Newton-Kantorovich's method. They construct a practical process based on the inverse way of the classical strategy. They start with the linearization phase of equation (1) and then go to the discretization phase. This (L.D) process has confirmed the high efficiency compared with the classical strategy (D.L) by theoretical and numerical results given in Grammont et al. [11–13] and recently in Khellaf et al. [14,15]. However, there are several works concerning elliptic PDEs such that the method used is consistent with the (D.L) strategy (see Gavete et al. [16]; Weiser et al. [17]).

This paper uses the same developed process as in Grammont et al. [11]; Khellaf et al. [15], but with a different discretization method. We start with the linearization phase and propose a double discretization phase, where this process will be denoted as (L.D.D), where it is called sometimes by" outer-inner iteration" process. This paper aims to create an (L.D.D) process that gives more smoothness and precision concerning the calculation part in the programming stage during the resolution procedure. The scheme (L.D.D) was introduced recently for the first time in Khellaf et al. [15], where the authors apply Sloan's method for the discretization phase. In this work, we apply Kantoroviche's method for the discretization phase, which gives us a different theoretical framework

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and shows fast convergence compared to Khellaf et al. [15]. In addition, the authors of paper Ahues et al. [18] propose a similar (L.D.D) scheme to solve nonlinear weakly singular integral equations.

The paper is organized as follows: In section 2, we present the main theoretical convergence results in which, under suitable assumptions on the (L.D.D) process, the sequence constructed converges to the exact solution of the main problem. In section 3 and 4, we illustrate these results with a numerical study and applications showing the accuracy and efficiency of our algorithms.

Theoretical results of convergence study

Let χ be a complex Banach space, where its norm is denoted by $\|\cdot\|$. The space $\mathcal{L}(\chi)$ defines the Banach algebra of bounded linear operators from χ to itself, where its norm is given by:

$$\forall A \in \mathcal{L}(\chi), \qquad \|A\| = \sup\{\|Ax\| : \ \|x\| \le 1\}.$$

Let assume that the main problem (1), has a unique solution, i.e.

(H_1) $\exists \psi \in \mathcal{V}: \ F(\psi) = 0_\chi$.

In general, to solve such a problem, we attempt to follow the Newton-type method applied in a finite-dimensional space. So, by linearizing equation (1), the exact solution is characterized as a limit of a sequence $(\psi^{(k)})_{k\ge 0}$ which is given through the following scheme:

$$F'(\psi^{(k)})(\psi^{(k+1)} - \psi^{(k)}) = -F(\psi^{(k)}), \quad \psi^{(0)} \in \mathcal{V}, \quad k = 1, 2, \cdots. \tag{2}$$

These iterated equations define the linearization phase. At this point, the difficulty of dealing with the exact formulate of $F'(\psi^{(k)})^{-1}$ becomes a major stumbling block in each iteration, because we are dealing with infinite-dimensional functional nonlinear equations. For this reason, we assume that.

(H_2) $F'(\psi)^{-1}$ exist, and $\exists \eta > 0$ such that $\left\|F'(\psi)^{-1}\right\| \le \eta < \infty$.

The convergence of Newton's method can be found in the book of Argyros et al. [19]. On the other hand, we can not compute $F'(\psi^{(k)})^{-1}$ exactly in each iteration. Thus we apply the discretization phase on scheme (2), so we define the following (L.D.D) iterate scheme, which is founded on a double discretization phase: Find $\psi_{n,m}^{(k+1)} \in \chi$, where $n, m \in \mathbb{N}$

$$F'_n(\pi_m \psi_{n,m}^{(k)})(\psi_{n,m}^{(k+1)} - \psi_{n,m}^{(k)}) = -F(\pi_m \psi_{n,m}^{(k)}), \quad \psi_{n,m}^{(0)} \in \mathcal{V}, \quad k = 1, 2\cdots. \tag{3}$$

The first discretization phase is applied on the operator $F'(x)$, which is approximated by $F'_n(x)$, and the second phase discretization is defined by involving the operator projection $(\pi_m)_{m\ge1}$ defined from χ into itself, on $F(\cdot)$ and $F'_n(\cdot)$ where, this projection satisfies the condition:

$$\forall v \in \chi, \quad \pi_m v \to v, \quad m \to \infty.$$

In the following, we show that the sequence $(\psi_{n,m}^{(k)})_{k\ge0}$ defined by problem (3), converges to ψ, the solution of equation (1), for k tends to infinity and the integers n and m are fixed. Let be $B_R(\psi)$ the ball of center ψ and radius $R > 0$. We assume that

(H_3) $F' : \mathcal{V} \to \mathcal{L}(\chi)$ is $\lambda -$ Lipschitz over $B_R(\psi)$.

We define the constant r such that,

$r := \min\{R, \frac{1}{2\eta\lambda}\}$.

For n large enough, we suppose also that the discretization process satisfies the following condition,

(H_4) $\forall x \in \mathcal{V} \subset \chi, \quad \left\|F'_n(x) - F'(x)\right\| \le \delta_n \to 0$, as $n \to \infty$.

The condition (H_4), is a sufficient condition to ensure the convergence of scheme (3). This condition is not satisfied by Sloan's method (See, Ahues et al. [20], page 187) which is established in Khellaf et al. [15] for an (L.D.D) scheme. However, we will show in the next section that Kantorovich's method will be holds with the condition (H_4).

Before starting the study of the convergence, we establish a series of lemmas that will be used in the proof of our convergence theorem.

lemma 2.1. *If the hypotheses $(H_1) - (H_3)$ are satisfied, then for all $x \in B_r(\psi)$, $F'(x)$ is invertible such that $\left\|F'(x)^{-1}\right\| \le 2\eta$.*

Proof 1. Let $x \in \mathcal{V} \subset \chi$, then $F'(x) = F'(\psi)\Big(I - F'(\psi)^{-1}[F'(\psi) - F'(x)]\Big)$, thus

$$\|F'(\psi)^{-1}[F'(\psi) - F'(x)]\| \le \|F'(\psi)^{-1}\| \, \|F'(\psi) - F'(x)\|.$$

Now, according to hypothesis (H_3), for all $x \in B_r(\psi)$, $\|F'(\psi) - F'(x)\| \le \lambda r$. Hence

$$\|F'(\psi)^{-1}(F'(\psi) - F'(x))\| \le \eta\lambda r \le \frac{1}{2}.$$

So, using the Geometric Series Theorem (see Atkinson et al. [1]), we conclude that $F'(x)$ is invertible, such that $F'(x)^{-1} = \Big(I - F'(\psi)^{-1}[F'(\psi) - F'(x)]\Big)^{-1} F'(\psi)^{-1}$. In addition, we find that

$$\begin{aligned} \|F'(x)^{-1}\| &= \left\|\big(I - F'(\psi)^{-1}[F'(\psi) - F'(x)]\big)^{-1} F'(\psi)^{-1}\right\| \\ &\le \|F'(\psi)^{-1}\| \left\|\big(I - F'(\psi)^{-1}[F'(\psi) - F'(x)]\big)^{-1}\right\| \\ &\le \eta \sum_{p=0}^{\infty} \|F'(\psi)^{-1}(F'(\psi) - F'(x))\|^p \\ &\le \eta \sum_{p=0}^{\infty} \left(\frac{1}{2}\right)^p = 2\eta, \end{aligned}$$

which completes the proof.

lemma 2.2. Assume that $(H_1) - (H_4)$ holds, then for all $x \in B_r(\psi)$ and for n large enough, the operator $F'_n(x)$ is invertible such that, $\sup_{x\in B_r(\psi)} \left\|I - F'_n(x)^{-1} F'(x)\right\| \le \rho_n$, $\sup_{x\in B_r(\psi)} \left\|F'_n(x)^{-1}\right\| \le 2\eta(1 + \rho_n)$,

where $\rho_n \to 0$ as $n \to \infty$.

Proof 2. For all $x \in B_r(\psi)$, and for $n \ge 1$, then by Lemma 2.1 $F'_n(x) = F'(x)\Big(I - F'(x)^{-1}\big[F'(x) - F'_n(x)\big]\Big)$. So, according to the hypothesis (H_4), such that $\delta_n < \frac{1}{2\eta}$, we find that $\left\|F'(x)^{-1}\big(F'(x) - F'_n(x)\big)\right\| \le 2\eta\delta_n < 1$. Now, using the Geometric Series Theorem, we conclude that $F'_n(x)$ is invertible, and $\left\|F'_n(x)^{-1}\right\| \le \frac{2\eta}{1-2\eta\delta_n}$. On the other hand, we notice that $I - F'_n(x)^{-1}F'(x) = F'_n(x)^{-1}\big(F'_n(x) - F'(x)\big)$. So, we define the sequence $\rho_n = \frac{2\eta\delta_n}{1-2\eta\delta_n}$. Thus, $\sup_{x\in B_r(\psi)} \left\|I - F'_n(x)^{-1}F'(x)\right\| \le \rho_n$. Similarly, we find the estimation, $\left\|F'_n(x)^{-1}\right\| \le 2\eta + 2\eta\rho_n$. This completes the proof.

lemma 2.3. Let $L(\cdot)$: $\chi \to \chi$ be *a* α-Lipschitz operator. For all $x \in \chi$, if $L(x)^{-1} \in \mathcal{L}(\chi)$ such that $\left\|L(x)^{-1}\right\| \le \mu$, then, $L^{-1}(\cdot)$ is $(\mu^2\alpha) -$ Lipschitz.

Proof 3. For all $x, y \in \chi$, we have, $L(x)^{-1} - L(y)^{-1} = L(x)^{-1}(L(y) - L(x))L(y)^{-1}$, hence

$$\|L(x)^{-1} - L(y)^{-1}\| \le \|L(x)^{-1}\| \|L(y) - L(x)\| \|L(y)^{-1}\| \le \mu^2\alpha\|x - y\|.$$

Proposition 2.1. Assume that $(H_1) - (H_4)$ holds. Then, for n large enough

$$F'_n(\cdot)^{-1} \text{is} (2+\lambda)(2\eta(1+\rho_n))^2 - \text{Lipschitz for all} x \in B_r(\psi).$$

Proof 4. For all $x, y \in B_r(\psi)$, then

$$F'_n(x) - F'_n(y) = \left(F'_n(x) - F'(x)\right) + F'(x) + \left(F'(y) - F'_n(y)\right) - F'(y).$$

So, using (H_3) and (H_4), we find that

$$\begin{aligned}\|F'_n(x) - F'_n(y)\| &\leq \|F'_n(x) - F'(x)\| + \|F'_n(y) - F'(y)\| + \|F'(x) - F'(y)\| \\ &\leq \delta_n + \delta_n + \lambda\|x - y\| \\ &\leq 2\,\delta_n + \lambda\|x - y\|\end{aligned}$$

Now, it is clear that, for n large enough such that $x \neq y$, we choose $\delta_n \leq \|x - y\|$. Thus

$$\|F'_n(x) - F'_n(y)\| \leq (2+\lambda)\|x - y\|.$$

So, we conclude $F'_n(x)$ is $(2 + \lambda)$- Lipschitz, and according to Lemma 2.2 we have

$$\|F'_n(x)^{-1}\| \leq 2\eta(1+\rho_n)$$

and according Lemma 2.3, we find that

$$F'_n(\cdot)^{-1}\text{is}(2\eta(1+\rho_n))^2(2+\lambda) - \text{Lipschitz}.$$

Thus, we have completed the proof.

Let the constants η, R, λ, ρ_n be defined in previous lemmas. We define also the constant ℓ such that,

$$\forall x, y \in \mathcal{V}: \quad \|F(x) - F(y)\| \leq \ell\|x - y\|.$$

The next theorem, is the principal result of our paper.

Theorem 2.1. If the initial function $\psi^{(0)}_{n,m} \in B_{\omega_n}(\psi)$, for $n, m \in \mathbb{N}$. Then the sequence $(\psi^{(k)}_{n,m})_{k\geq 0}$ defined by the scheme (3), converges to ψ the solution of equation (1), such that

$$\left\|\psi^{(k)}_{n,m} - \psi\right\| \leq \omega_n\left(\frac{1+\rho_n}{2}\right)^k \overset{k\to\infty}{\to} 0,$$

where,

$$\omega_n := \min\left\{\frac{r}{2}, \frac{1-\rho_n-4\eta\ell(1+\rho_n)}{2\left[\lambda\eta(1+\rho_n)+\ell(2+\lambda)(2\eta(1+\rho_n))^2\right]}\right\}.$$

Proof 5. Let $n, m \in \mathbb{N}$. If $\psi^{(0)}_{n,m} \in B_{\omega_n}(\psi)$ then according to Lemma 2.2, the operator $F'_n(\psi^{(0)}_{n,m})$ is invertible. Now by induction we assume that $\psi^{(k)}_{n,m} \in B_{\omega_n}(\psi)$. So, we have according to scheme (3),

$$\begin{aligned}\psi^{(k+1)}_{n,m} - \psi^{(k)}_{n,m} &= -F'_n(\pi_m\psi^{(k)}_{n,m})^{-1}F(\pi_m\psi^{(k)}_{n,m}) \\ &= -F'_n(\pi_m\psi^{(k)}_{n,m})^{-1}F\left(\psi^{(k)}_{n,m}\right) - F'_n(\pi_m\psi^{(k)}_{n,m})^{-1}\left[F(\pi_m\psi^{(k)}_{n,m}) - F(\psi^{(k)}_{n,m})\right] \\ &= -\left[F'_n(\pi_m\psi^{(k)}_{n,m})^{-1} - F'_n(\psi^{(k)}_{n,m})^{-1}\right]F(\psi^{(k)}_{n,m}) - F'_n(\psi^{(k)}_{n,m})^{-1}F(\psi^{(k)}_{n,m}) \\ &\quad -F'_n(\pi_m\psi^{(k)}_{n,m})^{-1}\left[F(\pi_m\psi^{(k)}_{n,m}) - F(\psi^{(k)}_{n,m})\right].\end{aligned}$$

Hence,

$$\begin{aligned}\psi^{(k+1)}_{n,m} - \psi = {} & \psi^{(k)}_{n,m} - \psi - F'_n(\psi^{(k)}_{n,m})^{-1}\left[F(\psi^{(k)}_{n,m}) - F(\psi)\right] \\ & -\left[F'_n(\pi_m\psi^{(k)}_{n,m})^{-1} - F'_n(\psi^{(k)}_{n,m})^{-1}\right]\left[F(\psi^{(k)}_{n,m}) - F(\psi)\right] \\ & -F'_n(\pi_m\psi^{(k)}_{n,m})^{-1}\left[F(\pi_m\psi^{(k)}_{n,m}) - F(\psi^{(k)}_{n,m})\right].\end{aligned}$$

So,

$$\begin{aligned}\left\|\psi^{(k+1)}_{n,m} - \psi\right\| \leq {} & \left\|\psi^{(k)}_{n,m} - \psi - F'_n(\psi^{(k)}_{n,m})^{-1}\left[F(\psi^{(k)}_{n,m}) - F(\psi)\right]\right\| \\ & + \left\|\left[F'_n(\pi_m\psi^{(k)}_{n,m})^{-1} - F'_n(\psi^{(k)}_{n,m})^{-1}\right]\left[F(\psi^{(k)}_{n,m}) - F(\psi)\right]\right\| \\ & + \left\|F'_n(\pi_m\psi^{(k)}_{n,m})^{-1}\left[F(\pi_m\psi^{(k)}_{n,m}) - F(\psi^{(k)}_{n,m})\right]\right\|.\end{aligned}$$

In this step, we estimate each part of this inequality separately. For the first part, we use the integral form of Lagrange's mean value formula (See Zarbrejko et al. [21] as follows:

$$\begin{aligned}\psi^{(k)}_{n,m} - \psi - F'_n(\psi^{(k)}_{n,m})^{-1}\left[F(\psi^{(k)}_{n,m}) - F(\psi)\right] &= \int_0^1\left[I - F'_n(\psi^{(k)}_{n,m})^{-1}F'((1-x)\psi^{(k)}_{n,m} + x\psi)\right]\left(\psi^{(k)}_{n,m} - \psi\right)dx \\ &= \int_0^1\left[I - F'_n(\psi^{(k)}_{n,m})^{-1}F'(\psi^{(k)}_{n,m})\right]\left(\psi^{(k)}_{n,m} - \psi\right)dx \\ &\quad - \int_0^1 F'_n(\psi^{(k)}_{n,m})^{-1}\left[F'((1-x)\psi^{(k)}_{n,m} + x\psi) - F'(\psi^{(k)}_{n,m})\right]\left(\psi^{(k)}_{n,m} - \psi\right)dx,\end{aligned}$$

and

$$\begin{aligned}&\left\|\psi^{(k)}_{n,m} - \psi - F'_n(\psi^{(k)}_{n,m})^{-1}\left[F(\psi^{(k)}_{n,m}) - F(\psi)\right]\right\| \leq \left\|I - F'_n(\psi^{(k)}_{n,m})^{-1}F'(\psi^{(k)}_{n,m})\right\|\left\|\psi^{(k)}_{n,m} \right. \\ &\left. - \psi\right\| + \left\|F'_n(\psi^{(k)}_{n,m})^{-1}\right\|\left\|\psi^{(k)}_{n,m} - \psi\right\|\int_0^1\left\|F'((1-x)\psi^{(k)}_{n,m} + x\psi) - F'(\psi^{(k)}_{n,m})\right\|dx.\end{aligned}$$

Now, since $\psi^{(k)}_{n,m} \in B_{\omega_n}(\psi)$, and according to Lemme 2.2, we find that

$$\left\|I - F'_n(\psi^{(k)}_{n,m})^{-1}F'(\psi^{(k)}_{n,m})\right\| \leq \rho_n,$$

and given that the ball $B_{\omega_n}(\psi)$ is a convex set, so for $x \in [0,1], (1-x)\psi^{(k)}_{n,m} + x\psi \in B_{\omega_n}(\psi)$. We use (H_3) to get

$$\left\|F'((1-x)\psi^{(k)}_{n,m} + x\psi) - F'(\psi^{(k)}_{n,m})\right\| \leq \lambda x\left\|\psi^{(k)}_{n,m} - \psi\right\|,$$

hence

$$\int_0^1\left\|F'((1-x)\psi^{(k)}_{n,m} + x\psi) - F'(\psi^{(k)}_{n,m})\right\|dx \leq \frac{1}{2}\lambda\left\|\psi^{(k)}_{n,m} - \psi\right\|.$$

So, we gather the first estimation as:

$$\left\|\psi_{n,m}^{(k)}-\psi-F_n'(\psi_{n,m}^{(k)})^{-1}\left[F(\psi_{n,m}^{(k)})-F(\psi)\right]\right\|\leq\rho_n\left\|\psi_{n,m}^{(k)}-\psi\right\|$$
$$+\left(2\eta(1+\rho_n)\left\|\psi_{n,m}^{(k)}-\psi\right\|\right)\frac{1}{2}\lambda\left\|\psi_{n,m}^{(k)}-\psi\right\|.$$

For the second part, as F is a Fréchet differentiable operator, there exist $\ell>0$ such that

$$\left\|F(\psi_{n,m}^{(k)})-F(\psi)\right\|\leq\ell\left\|\psi_{n,m}^{(k)}-\psi\right\|,$$

and according to Proposition 2.1 we have

$$\left\|\left[F_n'(\pi_m\psi_{n,m}^{(k)})^{-1}-F_n'(\psi_{n,m}^{(k)})^{-1}\right]\right\|\leq(2+\lambda)(2\eta(1+\rho_n))^2\left\|\pi_m\psi_{n,m}^{(k)}-\psi_{n,m}^{(k)}\right\|,$$

hence

$$\left\|\left[F_n'(\pi_m\psi_{n,m}^{(k)})^{-1}-F_n'(\psi_{n,m}^{(k)})^{-1}\right]\left[F(\psi_{n,m}^{(k)})-F(\psi)\right]\right\|\leq$$
$$\ell(2+\lambda)(2\eta(1+\rho_n))^2\left\|\psi_{n,m}^{(k)}-\psi\right\|\left\|\pi_m\psi_{n,m}^{(k)}-\psi_{n,m}^{(k)}\right\|.$$

For the third part, as

$$\left\|F(\pi_m\psi_{n,m}^{(k)})-F(\psi_{n,m}^{(k)})\right\|\leq\ell\left\|\pi_m\psi_{n,m}^{(k)}-\psi_{n,m}^{(k)}\right\|.$$

and $\psi_{n,m}^{(k)}\in B_{\omega_n}(\psi)$, so using the convergence of π_m to the identity operator, we find that.

$$\pi_m\psi_{n,m}^{(k)}\overset{m\to\infty}{\to}\psi_{n,m}^{(k)}\Leftrightarrow\forall\epsilon>0,\exists m_0\in\mathbb{N}^*,\forall m>m_0,\left\|\pi_m\psi_{n,m}^{(k)}-\psi_{n,m}^{(k)}\right\|\leq\epsilon,$$

Now, we choose m large enough, where if we put $\epsilon=\left\|\psi_{n,m}^{(k)}-\psi\right\|<\frac{r}{2}$, we get

$$\left\|\pi_m\psi_{n,m}^{(k)}-\psi\right\|\leq\left\|\pi_m\psi_{n,m}^{(k)}-\psi_{n,m}^{(k)}\right\|+\left\|\psi_{n,m}^{(k)}-\psi\right\|\leq2\left\|\psi_{n,m}^{(k)}-\psi\right\|\leq r,$$

So, we conclude that $\pi_m\psi_{n,m}^{(k)}\in B_{\omega_n}(\psi)$, and according to Lemma 2.2

$$\left\|F_n'(\pi_m\psi_{n,m}^{(k)})^{-1}\right\|\leq2\eta(1+\rho_n).$$

Then,

$$\left\|F_n'(\pi_m\psi_{n,m}^{(k)})^{-1}F(\pi_m\psi_{n,m}^{(k)})-F(\psi_{n,m}^{(k)})\right\|\leq2\eta(1+\rho_n)\ell\left\|\pi_m\psi_{n,m}^{(k)}-\psi_{n,m}^{(k)}\right\|.$$

Hence,

$$\left\|\psi_{n,m}^{(k+1)}-\psi\right\|\leq\rho_n\left\|\psi_{n,m}^{(k)}-\psi\right\|+\left(\lambda\eta(1+\rho_n)\left\|\psi_{n,m}^{(k)}-\psi\right\|\right)\left\|\psi_{n,m}^{(k)}-\psi\right\|$$
$$+\ell(2+\lambda)(2\eta(1+\rho_n))^2\left\|\psi_{n,m}^{(k)}-\psi\right\|\left\|\pi_m\psi_{n,m}^{(k)}-\psi_{n,m}^{(k)}\right\|$$
$$+2\eta\ell(1+\rho_n)\left\|\pi_m\psi_{n,m}^{(k)}-\psi_{n,m}^{(k)}\right\|,$$

and with more simplification, we find that

$$\left\|\psi_{n,m}^{(k+1)}-\psi\right\|\leq\left(\left[\rho_n+2\eta\ell(1+\rho_n)\right]+\left[\lambda\eta(1+\rho_n)\right.\right.$$
$$\left.\left.+\ell(2+\lambda)(2\eta(1+\rho_n))^2\right]\left\|\psi_{n,m}^{(k)}-\psi\right\|\right)\left\|\psi_{n,m}^{(k)}-\psi\right\|.$$

Now, we define the constant ω_n such that

$$\omega_n:=\min\left\{\frac{r}{2},\frac{1-\rho_n-4\eta\ell(1+\rho_n)}{2\left[\lambda\eta(1+\rho_n)+\ell(2+\lambda)(2\eta(1+\rho_n))^2\right]}\right\}.$$

So, as $\psi_{n,m}^{(k)}\in B_{\omega_n}(\psi)$, we obtain that

$$\left[\lambda\eta(1+\rho_n)+\ell(2+\lambda)(2\eta(1+\rho_n))^2\right]\left\|\psi_{n,m}^{(k)}-\psi\right\|\leq\frac{1-\rho_n-4\eta\ell(1+\rho_n)}{2},$$

hence

$$\left\|\psi_{n,m}^{(k+1)}-\psi\right\|\leq\left(\frac{1+\rho_n}{2}\right)\left\|\psi_{n,m}^{(k)}-\psi\right\|.$$

As $\frac{1+\rho_n}{2}<1$ that gives $\psi_{n,m}^{(k+1)}\in B_{\omega_n}(\psi)$. Finally we get the desired result

$$\left\|\psi_{n,m}^{(k)}-\psi\right\|\leq\omega_n\left(\frac{1+\rho_n}{2}\right)^k\overset{k\to\infty}{\to}0.$$

This completes the proof.

The following section will confirm our theoretical results by applying the (L.D.D) scheme with a double Kantorovich's discretization to solve nonlinear Fredholm integral equations. We selected Kantrovich's method because it assures a fast convergence process and satisfies the condition (H_4), unlike the case of the (L.D.D) scheme with a double Sloan's method (*See*, Ahues et al. [20] page 187).

Remark 1. Theoretically, the (L.D.D) new scheme is batter than the (D.L) classical method in the sense that $\psi_{n,m}^{(k)}\to\psi$ as $k\to+\infty$ whatever n and m big enough for the (L.D.D) scheme, unlike in the (D.L) classical method, where $\psi_n^{(k)}\to\psi$ as $k\to+\infty$ and $n\to+\infty$.

Application on nonlinear Fredholm integral equations using Kantorovich's projection

In this section, we will show how we use the hypotheses $(H_1)-(H_4)$ of Theorem 2.1 on the numerical examples provided. Let χ be the Banach space of real continued functions defined [0, 1]. Let K: $\chi\to\chi$ be a nonlinear integral compact operator defined as:

$$K(x)(t)=\int_0^1\kappa(t,s,x(s))\,ds,\quad x\in\mathcal{V},\quad t\in[0,1],$$

where the kernel κ is regular such that the operator K is Fréchet differentiable. We denote by $D=K'$ the Fréchet derivative of K such that

$$\forall x\in\chi,[D(x)v](t)=\int_0^1\frac{\partial\kappa}{\partial x}(t,s,x(s))v(s)\,ds,\quad v\in\chi,\quad t\in[0,1].$$

We set our problem as follows

$$\text{Find}\quad\psi\in\mathcal{V},\quad\psi(t)=\int_0^1\kappa(t,s,\psi(s))\,ds+g(t)\quad t\in[0,1],\tag{4}$$

Table 1
Numerical results of example 1, where we compared between our (L.D.D) scheme applying Kantorovich method and Khellaf et al. [15]-(L.D.D) scheme applying Sloan's method and the (D.L) classical method.

The error $E_{n,m}$ if $z=0.1$, $n=10$ and $m=10$							
k	(L.D.D)-Sloan of Khellaf et al. [15]	k	(L.D.D) Kantorovich	CPU time	k	(D.L) Classical	CPU time
$k=02$	2.20032 e-01	$k=02$	8.15484 e-01	5.2 e-02s	$k=02$	9.25584 e-01	8.1 e-03s
$k=06$	5.11577 e-03	$k=03$	4.66889 e-02	1.1 e-01s	$k=03$	5.69889 e-02	2.4 e-02s
$k=10$	1.71509 e-04	$k=04$	8.62274 e-05	1.7 e-01s	$k=04$	5.69889 e-02	7.3 e-02s
$k=14$	5.49420 e-05	$k=05$	8.97570 e-08	2.4 e-01s	$k=05$	5.69889 e-02	1.2 e-01s
$k=18$	5.59317 e-05	$k=06$	8.80032 e-08	3.1 e-01s	$k=06$	5.69889 e-02	1.5 e-01s

for a given function $g \in \chi$, and we can rewrite this problem as

$$\text{Find} \quad \psi \in \chi, \quad \psi = K(\psi) + g. \tag{5}$$

Let $\widetilde{e}_n = [e_1, \ldots, e_n] \in \chi^n$ be an order basis, where $\widetilde{e}_n^* = [e_1^*, \ldots, e_n^*] \in (\chi^*)^n$ is the adjoint order basis of $\widetilde{e}_n$. We define the operator projection π_n:

$$\pi_n x = \sum_{j=1}^{n} \langle x, e_j^* \rangle e_j,$$

where $\langle \cdot, \cdot \rangle$ is the duality brackets between χ and its dual space χ^*. Now, we define some of the notations in advance to use them to clarify the description of matrices and linear combinations:

$\widetilde{e}_n X = \sum_{j=1}^{n} X(j) e_j$ for all $X \in \mathbb{C}^n$,

$\ll V, \widetilde{e}_n^* \gg (i,j) = < v_j, e_i^* >$ for all $V = [v_1, v_2, \ldots, v_m] \in \chi^{1 \times m}$.

With the previous notation, we can write

$$\pi_n x = \widetilde{e}_n \ll x, \widetilde{e}_n^* \gg, \qquad x \in \chi.$$

As we defined in the last section, by applying the (L.D.D) process to solve problems like problem (5), we begin by the linearization phase using the Newton scheme to get the linear operator equation as follows:

$$\left(I - D(\psi^{(k)})\right)\left(\psi^{(k+1)} - \psi^{(k)}\right) = -F\left(\psi^{(k)}\right), \qquad \psi^{(k)} \in \mathcal{V}, \quad k = 1, 2, \ldots. \tag{6}$$

Next, for $n, m \in \mathbb{N}^*$, we apply a Double discretization Kantorovich projection to get our discretized linear problem: Find $\psi_{n,m}^{(k+1)} \in \chi$

$$\begin{cases} \psi_{n,m}^{(k+1)} - \pi_n D(\pi_m \psi_{n,m}^{(k)}) \psi_{n,m}^{(k+1)} = S_{n,m}^{(k)}, \\ S_{n,m}^{(k)} = K(\pi_m \psi_{n,m}^{(k)}) - \pi_n D(\pi_m \psi_{n,m}^{(k)}) \psi_{n,m}^{(k)} + g. \end{cases} \tag{7}$$

We suppose that

$$\begin{cases} (i)\,\text{Problem (5) has a unique solution } \psi \in \mathcal{V}, \\ (ii)\,(I - D(\psi)) \text{ is invertible, } \exists \eta > 0, \left\|(I - D(\psi))^{-1}\right\| \leq \eta < \infty, \\ (iii)\,D : \mathcal{V} \to \mathcal{L}(\chi) \text{ is } \lambda-\text{Lipschitz over } B_R(\psi). \end{cases} \tag{8}$$

By the previous assumptions, we guarantee the fulfillment of the hypotheses $(H_1) - (H_3)$, and now we must prove that the rest of the hypotheses are confirmed to apply our convergence Theorem. We recall that the discretization process defined in problem (7) is based on the double approximation as follows:

$$\text{For all } n \gg 1, \quad D_n(x) := \pi_n D(x), \quad x \in \chi.$$

As in Lemma 2.1, we can prove that, for all $x \in B_r(\psi)$, $(I - D(x))$ is invertible and

$$\left\|(I - D(x))^{-1}\right\| \leq 2\eta.$$

Proposition 3.1. Assume that the hypotheses (8) hold, and we suppose also that

1. $\forall x \in \chi,\ \pi_n x \overset{n \to \infty}{\longrightarrow} x$,
2. The set

$$Z := \{D(x)z \mid x \in B_R(\psi),\ z \in \chi,\ \|z\| = 1\},$$

is relatively compact. Then

$$\sup_{x \in B_R(\psi)} \|D_n(x) - D(x)\| \leq \delta_{n,m} \overset{n,m \to \infty}{\longrightarrow} 0.$$

Proof 6. The set Z is a relatively compact set, so π_n tends to the identity operator I pointwess, thus we conclude that the pointwise convergence on relatively compact sets is a uniform convergence.

We note that the Kantorovich projection method verifies point 1) of Proposition 3.1 (see Ahues et al. [20] pp.185). Moreover, the integral operators are compact operators, so point 2) of the same proposition is well verified. By using Proposition 3.1, hypotheses (8) and according to Lemma 2.2 we can determine that the operator $(I - D_n(x))^{-1}$ is invertible and.

$$\sup_{x \in B_r(\psi)} \left\|(I - D_n(x))^{-1}\right\| \leq 2\eta(1 + 2\rho_n),$$

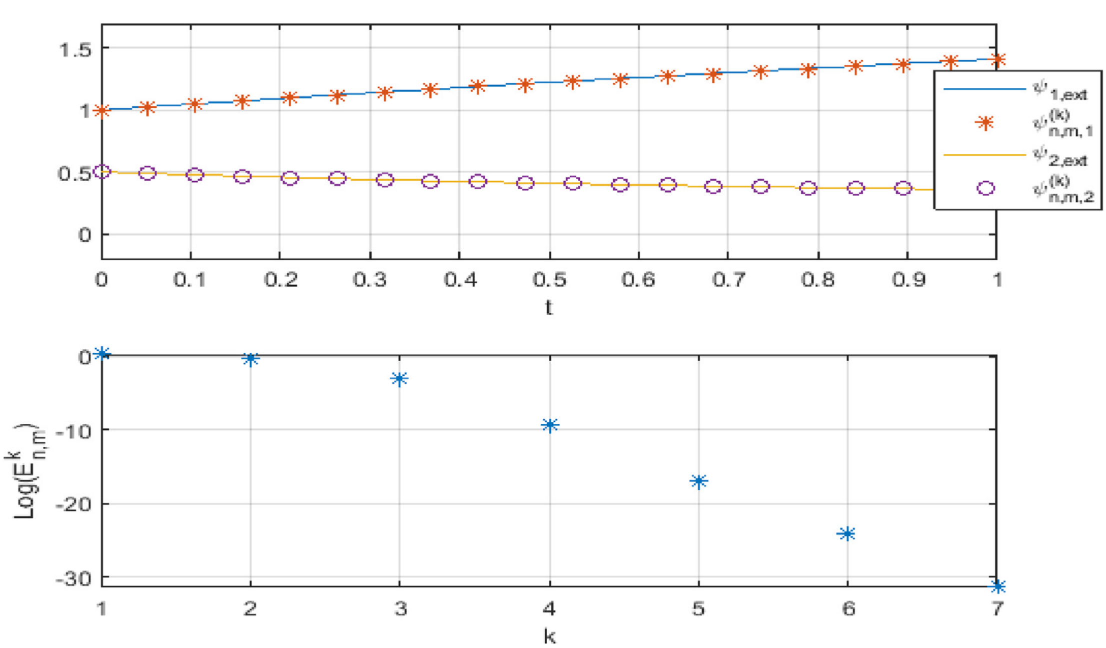

Fig. 1. Approximate solutions of example 1, using (L.D.D) Method.

where r, ρ_n are parameters defined in Lemmas 2.1 − 2.2. To this point, we have satisfied all the hypotheses of Theorem 2.1. Thus, we have guaranteed that

$$\psi^{(k)}_{n,m} \overset{k\to\infty}{\to} \psi, \quad \text{for } n,m \text{ fixed in } \mathbb{N}.$$

Let us concentrate on the implementation of our (L.D.D) scheme. We remark that,

$$S^{(k)}_{n,m} = K(\pi_m\psi^{(k)}_{n,m}) - \pi_n D(\pi_m\psi^{(k)}_{n,m})\psi^{(k)}_{n,m} + g,$$

and we can write,

$$(I-\pi_n)\psi^{(k+1)}_{n,m} = (I-\pi_n)S^{(k)}_{n,m} = (I-\pi_n)(K(\pi_m\psi^{(k)}_{n,m}) + g),$$

and accordingly

$$\psi^{(k+1)}_{n,m} = (I-\pi_n)(K(\pi_m\psi^{(k)}_{n,m}) + g) + \widetilde{e}_n U^{(k+1)}_{n,m}, \tag{9}$$

where $U^{(k+1)}_{n,m} \in \mathbb{C}^n$ is a column vector we get it by solving the following linear system

$$\left(I_n - M^{(k)}_n\right)U^{(k+1)}_{n,m} = d^{(k)}_{n,m},$$

where for $1 \le i, j \le n$, we have.

$$M^{(k)}_n(i,j) = \ll D(\pi_m\psi^{(k)}_{n,m})\widetilde{e}_n, \widetilde{e}^*_n \gg (i,j) = \langle D(\pi_m\psi^{(k)}_{n,m})e_j, e^*_i\rangle.$$
$$d^{(k)}_{n,m}(i) = \langle K(\pi_m\psi^{(k)}_{n,m}), e^*_i\rangle - \langle D(\pi_m\psi^{(k)}_{n,m})\psi^{(k)}_{n,m}, e^*_i\rangle + \langle g, e^*_i\rangle + \langle D(\pi_m\psi^{(k)}_{n,m})(I-\pi_n)(K(\pi_m\psi^{(k)}_{n,m}) + g), e^*_i\rangle.$$

The technique of creating a system of nonlinear Fredholm integro-differential equations

This subsection explains how to create a system of nonlinear Fredholm integro-differential equations starting from one equation. Moreover, rewrite this system in the same form as our main problem (5) to solve it by applying our (L.D.D) scheme. Consider the following integro-differential equation:

$$\psi(t) = \int_0^1 \kappa\left(t, s, \psi(s), \psi'(s), \ldots, \psi^{(N-1)}(s)\right)ds + g(t), \quad t \in [0,1], \tag{10}$$

For $t \in [0, 1]$, we derive this equation $(N-1)$ times to obtain the following system

$$\begin{cases} \psi(t) = \int_0^1 \kappa\left(t,s,\psi(s),\psi'(s),\ldots,\psi^{(N-1)}(s)\right)ds + g(t), \\ \psi'(t) = \int_0^1 \frac{\partial\kappa}{\partial t}\left(t,s,\psi(s),\psi'(s),\ldots,\psi^{(N-1)}(s)\right)ds + g'(t), \\ \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots \\ \psi^{(N-1)}(t) = \int_0^1 \frac{\partial^{(N-1)}\kappa}{\partial t^{(N-1)}}\left(t,s,\psi(s),\psi'(s),\ldots,\psi^{(N-1)}(s)\right)ds + g^{(N-1)}(t). \end{cases} \tag{11}$$

If we set, for all $1 \le i \le N$, $g_i(t) = g^{(i-1)}(t)$, $\psi_i(t) = \psi^{(i-1)}(t)$ and $\kappa_i = \frac{\partial^{(i-1)}\kappa}{\partial t^{(i-1)}}$, we get the following system

$$\begin{cases} \psi_1(t) = \psi(t) = \int_0^1 \kappa_1(t,s,\psi_1(s),\psi_2(s),\ldots,\psi_N(s))ds + g_1(t), \\ \psi_2(t) = \psi'(t) = \int_0^1 \kappa_2(t,s,\psi_1(s),\psi_2(s),\ldots,\psi_N(s))ds + g_2(t), \\ \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots \\ \psi_N(t) = \psi^{(N-1)}(t) = \int_0^1 \kappa_N(t,s,\psi_1(s),\psi_2(s),\ldots,\psi_N(s))ds + g_N(t). \end{cases} \tag{12}$$

This system can be rewritten as the ensuing structure

$$\text{Find } \psi \in \prod_{i=1}^{N} C^{(N-i)}([0,1],\mathbb{R}), \qquad \psi = K(\psi) + G. \tag{13}$$

where $\psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \vdots\\ \psi_N\end{pmatrix}, G = \begin{pmatrix}g_1\\ g_2\\ \vdots\\ g_N\end{pmatrix}, K = \begin{pmatrix}K_1\\ K_2\\ \vdots\\ K_N\end{pmatrix}$ and $K_i = \int_0^1 \kappa_i(t,s,\chi(s))ds$, and the problem (5) and (13) become the same, so we can apply our (L.D.D) scheme to solve the problem (13) as we will do in the numerical examples in the next section.

Numerical examples

In this section, we confirm the efficacy of our (L.D.D) scheme by solving three problems. In the first example, we will solve a system of nonlinear Fredholm integro-differential equations, which has been treated in Khellaf et al. [15], where it is solved by the (L.D.D) scheme using Sloan's discretization, what is more, we will compare the results obtained in Khellaf et al. [15], the (D.L)-Classical method and our results obtained by applying the (L.D.D) scheme applying Kantorovich's discretization. In the second and the third examples, we will treat a system of two and three (respectively) nonlinear Fredholm integro-differential using our (L.D.D) scheme and the (D.L)-Classical process. All results of numerical applications are summarized and offered in tables and figures separately.

First, let $n \in \mathbb{N}^*$, and considering the equidistant subdivision Δ_n of [0, 1] defined by:

$$\Delta_n = \left\{t_p = p\,h,\ h = \tfrac{1}{n}, p = 0, 1, \ldots, n\right\}.$$

Let $\left(\psi^{(k)}_{n,m,1}, \psi^{(k)}_{n,m,2}, \ldots, \psi^{(k)}_{n,m,N}\right) \in \mathcal{V}\subset\chi, k \in \mathbb{N}^*$ the k order approximative solution of our equations system (13) according to scheme (7) obtained bu apply the (L.D.D) method. We specify the stopping condition on the parameter k as:

$$E^k_{n,m} = \sum_{i=1}^{N} \max_{0\le p\le n}\left|\psi^{(k+1)}_{n,m,i}(t_p) - \psi^{(k)}_{n,m,i}(t_p)\right| \le 10^{-09}.$$

We denote the obtained error formula by

$$E_{n,m} = \sum_{i=1}^{N} \max_{0\le p\le n}\left|\psi_{i,ext}(t_p) - \psi^{(k)}_{n,m,i}(t_p)\right|,$$

Table 2
Numerical results of example 2.

The error $E_{n,m}$ if $z = 10$ at $k = 3$								
n	(L.D.D)-Kantorovich				(D.L)-Classical			
	$m = 50$	CPU time	$m = 100$	CPU time	$m = 50$	CPU time	$m = 100$	CPU time
5	6.6891 e-08	9.3 e-02s	1.6226 e-08	1.1 e-01s	1.0146 e-02	1.3 e-02s	1.0146 e-02	1.3 e-02s
20	4.1586 e-09	5.9 e-01s	8.5297 e-10	6.6 e-01s	1.0096 e-02	4.5 e-02s	1.0095 e-02	5.2 e-02s
50	1.1830 e-09	2.6 e-00s	1.8573 e-10	2.8 e-00s	1.0094 e-02	2.5 e-01s	1.0093 e-02	2.0 e-01s
100	1.0354 e-09	9.2 e-00s	5.9792 e-11	9.6 e-00s	1.0094 e-02	5.5 e-01s	1.0093 e-02	5.8 e-01s

where, $\Psi_{ext} = (\psi_{1,ext}, \psi_{2,ext}, \dots, \psi_{N,ext}) \in \mathcal{V} \subset \chi$ is the exact solution of the initial equations system (13). We pass directly to the numerical examples.

Example 1. Consider the following nonlinear Fredholm integro-differential equation

$$\psi(t) = z\int_0^1 \left(\exp(t)(\psi(s))^2 - \exp(-t)(\psi'(s))^2\right)ds + g(t), \quad z \in \mathbb{R}, t \in [0,1], \tag{14}$$

with $\psi \in C^1([0,1], \mathbb{R})$, and the function g is given by

$$g(t) = \sqrt{1+t} - \frac{z}{4}(6\exp(t) - \log(2)\exp(-t)), \quad t \in [0,1].$$

We derive equation (14) to obtain the following system of nonlinear integro-differential equations

$$\begin{cases} \psi(t) = z\int_0^1 \left(\exp(t)(\psi(s))^2 - \exp(-t)(\psi'(s))^2\right)ds + g(t), \\ \psi'(t) = z\int_0^1 \left(\exp(t)(\psi(s))^2 + \exp(-t)(\psi'(s))^2\right)ds + g'(t), \end{cases} \tag{15}$$

and the function g' is given by

$$g'(t) = \frac{1}{2\sqrt{1+t}} - \frac{z}{4}(6\exp(t) + \log(2)\exp(-t)), \quad t \in [0,1].$$

So, system (15) is similar to the following system

$$\begin{cases} \psi_1(t) = z\int_0^1 \left(\exp(t)(\psi_1(s))^2 - \exp(-t)(\psi_2(s))^2\right)ds + g_1(t), \\ \psi_2(t) = z\int_0^1 \left(\exp(t)(\psi_1(s))^2 + \exp(-t)(\psi_2(s))^2\right)ds + g_2(t), \end{cases} \tag{16}$$

where $\Psi_{ext} = \left(\sqrt{1+t}, \frac{1}{2\sqrt{1+t}}\right)$ is the exact solution of our system.

Algorithm 1: (L.D.D) Algorithm

```
Data: n, m, g, ψ_ext
Result: E_{n,m}, plot (ψ_{n,m}^{(k+1)}, ψ_ext, log(e))
Initialization: ψ_{n,m}^{(0)} ← 0, M_n^{(0)} ← 0_{n×n}, d_n^{(0)} ← 0_n, k ← 1, Tol ← 10^{-12}, E_{n,m}^k ← 1;
while E_{n,m}^k > Tol do
    for i ← 1 to n do
        for j ← 1 to n do
            Calculate and save M_n^{(k)}(i, j);
            if i=j then
                A_n^{(k)} ← 1 − M_n^{(k)}(i, j);
            else
                A_n^{(k)} ← −M_n^{(k)}(i, j);
            end
        end
        Calculate and save d_n^{(k)}(i);
    end
    X_n^{(k+1)} ← (A_n^{(k)})^{-1}.d_n^{(k)};   /* (Calculate and save the vector solution
     of the linear system) */
    for p ← 1 to n do
        π_n K(t) ← π_n K(t) + K(T_n(p)) . e_p(t);
        π_n g(t) ← π_n g(t) + g(T_n(p)) . e_p(t);
        e_n X_n^{(k+1)}(t) ← e_n X_n^{(k+1)}(t) + X_n^{(k+1)}(p) . e_p(t);
    end
    ψ_{n,m}^{(k+1)}(t) ← K(t) + g(t) − π_n K(t) − π_n g(t) + e_n X_n^{(k+1)}(t);
    ψ_{n,m}^{(k+1)}(t) ← ψ_{n,m}^{(k)}(t);     /* (Save the previous iterate solution, k =
     1,2...) */
    E_{n,m}^k(t) ← max_{t∈[0,1]} ‖ψ_{n,m}^{(k+1)}(t) − ψ_{n,m}^{(k)}(t))‖; /* (Calculate the iterate error)
     */
    k ← k + 1;      /* (Increment the number of iterations k by 1) */
end
E_{n,m} ← max_{t∈[0,1]} ‖ψ_{n,m}^{(k+1)}(t) − ψ_ext(t)‖ ;    /* (Calculate the error between the
 exact solution and the last iterate solution) */
```

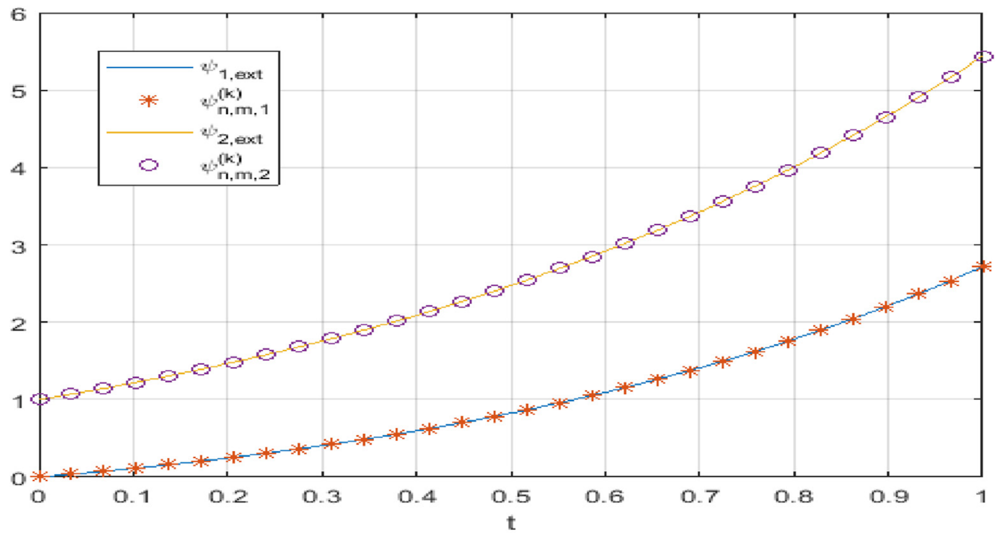

Fig. 2. Approximate solutions of example 2, using (L.D.D) Method.

Example 2. Consider the following nonlinear Fredholm integro-differential equation

$$\psi(t) = \frac{z}{20}\int_0^1 \cos\Big(\exp(s) + \arccos\Big(\frac{s+t}{3}\Big) + \psi(s) - \psi'(s)\Big)ds + g(t), \quad z \in \mathbb{R}, t \in [0,1], \tag{17}$$

with $\psi \in C^1([0,1],\mathbb{R})$, and the function g is given by

$$g(t) = t\exp(t) - \frac{z}{60}\Big(t + \frac{1}{2}\Big), \quad t \in [0,1].$$

We derive equation (17) to obtain the following system of nonlinear integro-differential equations

$$\begin{cases} \psi(t) = \dfrac{z}{20}\displaystyle\int_0^1 \cos\Big(\exp(s) + \arccos\Big(\frac{s+t}{3}\Big) + \psi(s) - \psi'(s)\Big)ds + g(t), \\ \psi'(t) = \dfrac{z}{60}\displaystyle\int_0^1 \frac{1}{\sqrt{1-\Big(\frac{s+t}{3}\Big)^2}} \sin\Big(\exp(s) + \arccos\Big(\frac{s+t}{3}\Big) + \psi(s) - \psi'(s)\Big)ds + g'(t), \end{cases} \tag{18}$$

and the function g' is given by

$$g'(t) = (1+t)\exp(t) - \frac{z}{60}, \quad t \in [0.1].$$

So, system (18) is similar to the following system

$$\begin{cases} \psi_1(t) = \dfrac{z}{20}\displaystyle\int_0^1 \cos\Big(\exp(s) + \arccos\Big(\frac{s+t}{3}\Big) + \psi_1(s) - \psi_2(s)\Big)ds + g_1(t), \\ \psi_2(t) = \dfrac{z}{60}\displaystyle\int_0^1 \frac{1}{\sqrt{1-\Big(\frac{s+t}{3}\Big)^2}} \sin\Big(\exp(s) + \arccos\Big(\frac{s+t}{3}\Big) + \psi_1(s) - \psi_2(s)\Big)ds + g_2(t), \end{cases} \tag{19}$$

where $\Psi_{ext} = (t\exp(t), (t+1)\exp(t))$ is the exact solution of our system.

Example 3. Consider the following nonlinear Fredholm integro-differential equation

$$\psi(t) = \frac{z}{50}\int_0^1 \frac{\sin(\pi s)}{1+t}(2\pi\psi(s) + \psi'(s)^2 - \cos(\pi s)\psi''(s))ds + g(t), \quad z \in \mathbb{R}, t \in [0,1], \tag{20}$$

with $\psi \in C^2([0,1],\mathbb{R})$, and the function g is given by

$$g(t) = \cos(\pi t) - \frac{z\pi}{25(1+t)}, \quad t \in [0,1].$$

We derive equation (20) two times to obtain the following system of nonlinear integro-differential equations

$$\begin{cases} \psi(t) = \dfrac{z}{50}\displaystyle\int_0^1 \dfrac{\sin(\pi s)}{1+t}\left(2\pi\psi(s) + \psi'(s)^2 - \cos(\pi s)\psi''(s)\right) ds + g(t), \\ \psi'(t) = -\dfrac{z}{50}\displaystyle\int_0^1 \dfrac{\sin(\pi s)}{(1+t)^2}\left(2\pi\psi(s) + \psi'(s)^2 - \cos(\pi s)\psi''(s)\right) ds + g'(t), \\ \psi''(t) = \dfrac{z}{25}\displaystyle\int_0^1 \dfrac{\sin(\pi s)}{(1+t)^3}\left(2\pi\psi(s) + \psi'(s)^2 - \cos(\pi s)\psi''(s)\right) ds + g''(t), \end{cases} \tag{21}$$

where $g'(t) = -\pi\sin(\pi t) + \frac{z\pi}{25(1+t)^2}$ and $g''(t) = -\pi^2\cos(\pi t) - \frac{2z\pi}{25(1+t)^3}$,

and by the same notation technique always do, system (21) is similar to the following system

$$\begin{cases} \psi_1(t) = \dfrac{z}{50}\displaystyle\int_0^1 \dfrac{\sin(\pi s)}{1+t}\left(2\pi\psi_1(s) + \psi_2(s)^2 - \cos(\pi s)\psi_3(s)\right) ds + g_1(t), \\ \psi_2(t) = -\dfrac{z}{50}\displaystyle\int_0^1 \dfrac{\sin(\pi s)}{(1+t)^2}\left(2\pi\psi_1(s) + \psi_2(s)^2 - \cos(\pi s)\psi_3(s)\right) ds + g_2(t), \\ \psi_3(t) = \dfrac{z}{25}\displaystyle\int_0^1 \dfrac{\sin(\pi s)}{(1+t)^3}\left(2\pi\psi_1(s) + \psi_2(s)^2 - \cos(\pi s)\psi_3(s)\right) ds + g_3(t), \end{cases} \tag{22}$$

where $\Psi_{ext} = (\cos(\pi t), -\pi\sin(\pi t), -\pi^2\cos(\pi t))$ is the exact solution of our system.

The results of (L.D.D)-Kantorovich's, (L.D.D)-Sloan's and (D.L)-Classical applied on example 1 are indicated in Table 1 and Fig. 1, which confirm that our process is more powerful based on the error value obtained, and its approximate solution converges in 6 iterations ($k = 6$), where it is faster than the (L.D.D)-Sloan's method that converges in 18 iterations ($k = 18$) and faster than the (D.L)-Classical method that do not get the convergence for n small. Table 2 shows that our (L.D.D)-Kantorovich's method gets better where we increase n and m, with a small number of iterations ($k = 3$) (See, Fig. 2) and the (D.L)-Classical process do not converge for a small n. Table 3 demonstrates that the best approximate solution can be obtained with a fixed n (at most 15) and the more we increase m (at most 30), the more we get a better approximate solution (See, Fig. 3), unlike the (D.L)-Classical method that needs to take n bigger than 1000 or more to get the convergence. These results correspond to the theoretical part that we presented in this research. However, in the three tables, we also compared the execution time of the (L.D.D)-Kantorovich's and (D.L)-Classical methods, and we got similar results, where we used Matlab computation software, with a machine of type Intel(R) Core(TM) i7-8665U CPU @1.90 GHz 2.11 GHz and 32 GB RAM.

Table 3
Numerical results of example 3.

The error $E_{n,m}$ if $z = 0.1$, $n = 15$ at $k = 5$				
m	(L.D.D)-Kantrovich	CPU time	(D.L)-Classical	CPU time
5	1.037871850698868 e-04	2.92s	7.7395458721862 e-02	1.65s
10	4.615256983570827 e-06	3.05s	7.7273067743482 e-02	1.72s
15	1.176507064125706 e-06	3.09s	7.7055432555865 e-02	1.79s
30	5.045708066493429 e-07	3.12s	7.6925458071442 e-02	1.85s

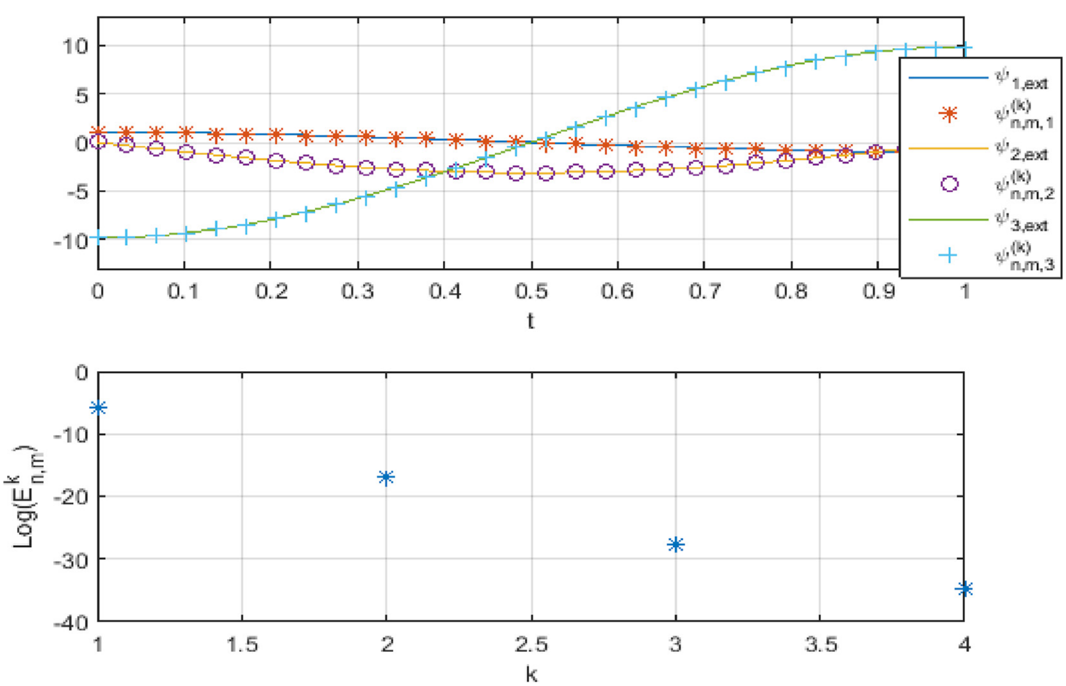

Fig. 3. Approximate solutions of example 3, using (L.D.D) Method.

Conclusion

This paper presents a new numerical scheme to treat nonlinear functional equations, specifically systems of nonlinear Fredholm integro-differential equations of the second kind. The (L.D.D) scheme begins by linearizing then double discretizing process, and it has proved its efficacy and precision in numerical applications. Generally, we intend to develop this scheme in future works to make it able to solve systems of nonlinear Fredholm integro-differential equations of the second kind with a weakly singular kernel.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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