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Numerical Solution of Non-linear Volterra Integral Equation of the First Kind

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Abstract: In this paper, we focus on the numerical solution of a nonlinear Volterra equation of the first kind. The existence and uniqueness of the exact solution are ensured under a necessary condition which we present next. We develop a numerical method based on two essential parts which are linearization and discretization. We start with the discretization of the equations using the concept of Nystrom's method and for the linearization we apply Newton's method. We present theorems that show the convergence of the proposed method. At the end, numerical examples are provided to show the efficiency of our method.

Key Words: Volterra integral equation, non-linear integral equation, numerical integration, Newton method.

Contents

1. Introduction

Volterra integral equations are the good choice for scientists to model many evolution problems. Historically, the emergence of this equation was to express population dynamics through Volterra's most famous work, as it constituted a qualitative leap in modeling and applied mathematics [15], and since then, the use of this type of equations have expanded year after year. In year 1976 [9], it was the best solution for modeling population evolution, in 1988 it was used to express vibrational motion and was applied to express the sorption kinetics of mixtures in 1990, then we find it a year 1996 in semi conductor devices and then to express viscosity in 1999. The above was a slight review of some of the works presented in the last century and to date it has countless applications like the tumor growth [10,12], modeling the system related to leukemia [13], birth-death process [11] and relativistic quantum physics [14].

Because these equations forms varied according to the domain where they are applied, there are different types and shapes. In this manuscript, we focus on one particular type, which is the non-linear Volterra equation. This has been presented in many works in the following general form:

$$\forall x \in [0, X], \int_0^x \mathcal{K}(x, t, u(t))\, dt = f(x),\ X < +\infty, \tag{1.1}$$

where f is a given function in the Banach space $C^1[0, X]$.

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The above equation first appeared in the work of Abel to express the motion of a particle along a series curve. For our part, we present some of the recent work in order to shed light on its great importance. The importance of this equation was not limited to a specific field or domain. We find it in engineering [1,3,4], in physics [6], in artificial neutwork [7], in meteorology to express the amount of precipitation [8], in medicine, where it has been used recently to study the evolution of the Corona virus or to find the best model of it [2,5], the population dynamics and spread of epidemics [16]. The equation (1.1) is considered to be one of the problems that are ill-posed, according to Hadamard's definition [20]. Therefore, we find many researches concerned with the development of the regularization methods [21,22]. On the other side, there are many scientific papers that deal with the search for a numerical solution to this type of equation, in which many numerical methods have been invented, and we mention among them: Petryshyn's fixed point theorem [24], homotopy perturbation [25], discrete operational vector scheme [25], hp-vertion collocation method [27] and Nyström method [28,29,30,31,32,33]. In this manuscript, we recall sufficient conditions to prove the existence and uniqueness of the solution in the Banach space $C^0[0,X]$. For the method approach, we develop a numerical method based on two principle ideas of Nyström and Newton procedures. Therefore, we use some concepts of linearization and discretization that have been applied in the treatment of the nonlinear Fredholm integral equation of the second kind [34,35,36].

2. Problem Position

In this paper, we are interested in developing a method for finding the best possible approximate solution. But, we do not deal directly with the equation (1.1) for the aforementioned reason. Instead, we suggest that the kernel $\mathcal{K}$ is derivable with respect to x. So, that we can convert the equation into a non-linear Volterra equation of second kind:

$$\forall x \in [0,X],\ \mathcal{K}(x,x,u(x)) = f'(x) - \int_0^x \frac{\partial \mathcal{K}}{\partial x}(x,t,u(t))\, dt. \tag{2.1}$$

In order to make the presentation clearer and easy to read, let $C^0[0,X]$ be a Banach space equipped with next norm

$$\forall v \ \in C^0[0,X],\ ||v||_{C^0[0,X]} = \max_{0\le x\le X} |u(x)|.$$

3. Analytical Study

Since we cannot invent a numerical solution without ensuring the existence and uniqueness of the exact solution, we assume that the kernel verifies the below hypotheses which are sufficient for ensuring this.

$$(H_1)\ \left\| \begin{array}{ll} (1) & f(0)=0, \\ (2) & f \text{ and } f' \text{ in } C^0[0,X], \\ (3) & \mathcal{K} \text{ and } \dfrac{\partial \mathcal{K}}{\partial x} \in C^0\Big([0,X]^2\times\mathbb{R}\Big), \\ (4) & \exists\ M>0,\ \max\limits_{0\le x,t\le X}\left(|\mathcal{K}(x,t,v)|, \left|\dfrac{\partial \mathcal{K}}{\partial x}(x,t,v)\right|\right) \le M, \\ (5) & \exists\ L>0,\ \forall\ x,t\in[0,X], \forall v,\bar{v}\in\mathbb{R}\text{: } \left|\dfrac{\partial \mathcal{K}}{\partial x}(x,t,v) - \dfrac{\partial \mathcal{K}}{\partial x}(x,t,\bar{v})\right| \le L\,|v-\bar{v}|, \\ (6) & \forall\ y\in\mathbb{R}, \exists\,!\ v : \mathcal{K}(x,x,v) = y, \\ (7) & \exists\ \theta>0, \forall\ x,t, \forall v,\bar{v}\in\mathbb{R} : |\mathcal{K}(x,t,v) - \mathcal{K}(x,t,\bar{v}) \ge \theta\ |v-\bar{v}|. \end{array}\right.$$

Let us apply the successive approximation method or Picard's method [16] to equation (2.1).This consists in the construction of sequences $\{u_n\}_{n\in\mathbb{N}}$ and $\{\psi_n\}_{n\in\mathbb{N}}$ which are defined by

$$\begin{cases} u_0(x) = f'(x), \\ \mathcal{K}(x, x, u_n(x)) = f'(x) - \displaystyle\int_0^x \frac{\partial \mathcal{K}}{\partial x}(x, t, u_{n-1}, (t))\, dt, \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

and

$$\begin{cases} \psi_0(x) = f'(x), \\ \psi_n(x) = u_n(x) - u_{n-1}(x), \quad \forall n \geq 1. \end{cases} \tag{3.2}$$

Theorem 3.1. *Under the hypotheses (H_1), the equation (2.1) has a unique solution in $C^0[0, X]$.*

Proof. Details can be found in [16]. □

4. Numerical Study

In the search for a numerical solution of the equation (1.1), we follow two essential steps: The first one is to discretize our problem (1.1) by replacing the integral sign by a numerical integration, whereas the second step is the linearization of the new approximate problem using Newton's method. In order to estimate the numerical error, we use the continuity module $\kappa_0(.,h)$ by

$$\forall\, v \in C^0[0, X], \kappa_0(v, h) = \sup_{|x-y|\leq h} |v(x) - v(y)|.$$

4.1. Discretization

We define Δ_n, $n \geq 1$ the uniform discretization of the interval $[0, X]$ as:

$$\Delta_n = \Big\{ n \geq 1, 0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = X,\ h = x_{j+1} - x_j,\ 0 \leq j \leq n \Big\}. \tag{4.1}$$

By choosing the collocation points $x = x_i$, we obtain

$$\mathcal{K}(x_i, x_i, u(x_i)) + \int_0^{x_i} \frac{\partial \mathcal{K}}{\partial x}(x_i, t, u(t))\, dt = f(x_i), \quad 0 \leq i \leq n. \tag{4.2}$$

Our main goal is searching u_i as an approximation of $u(x_i)$. So, we replace the integral sign in (4.2) by the following numerical Gaussian scheme:

$$\forall v \in C^0[0, X], \quad \int_0^{x_i} v(x)\, dx \approx h \sum_{j=0}^{i} \omega_j v(x_j), \quad 0 \leq i \leq n,$$

such that $\{\omega_i\}_{0\leq i\leq n}$ is called the weights and verify:

$$\exists\, W > 0, \sup_{0\leq i\leq n} |\omega_i| \leq W < +\infty,\ n \geq 1.$$

We get the following discrete problem: For all i fixed, find u_i solution of the equation:

$$\mathcal{K}(x_i, x_i, u_i) + h \sum_{j=0}^{i} \omega_j \frac{\partial \mathcal{K}}{\partial x}(x_i, x_j, u_j) = f'(x_i), \tag{4.3}$$

which is equivalent for all $i = 0, \ldots n$ and $n \geq 1$, to non-linear equation:

$$\begin{cases} \mathcal{K}(x_i, x_i, u_i) + h w_i \dfrac{\partial \mathcal{K}}{\partial x}(x_i, x_i, u_i) = S_i, \\ S_i = f'(x_i) - h \sum\limits_{j=0}^{i-1} \omega_j \dfrac{\partial \mathcal{K}}{\partial x}(x_i, x_j, u_j). \end{cases} \tag{4.4}$$

Before studying the convergence of approximate solution, we need to show that the system (4.4) has a unique solution.

Theorem 4.1. *For h sufficiently small, the system (4.4) has a unique solution.*

Proof. For $in \geq 1$ fixed, suppose that $\{u_i\}_{0\leq i\leq n}$ are known. Now, we present two sequences: the first is $\{u_i^p\}_{p\in\mathbb{N}}$ and verifies the problem

$$\begin{cases} \mathcal{K}(x_i, x_i, u_i^{p+1}) = S_i - hw_i \dfrac{\partial \mathcal{K}}{\partial x}(x_i, x_i, u_i^p), \quad p \geq 1, \\ \\ u_i^0 = u_{i-1}, \end{cases} \tag{4.5}$$

and the second is $\{\phi_i^p\}_{p\in\mathbb{N}}$ and is defined by

$$\begin{cases} \phi_i^p &= u_i^p - u_i^{p-1}, \quad p \geq 1, \\ \phi_i^0 &= u_{i-1}. \end{cases} \tag{4.6}$$

It is clear that $\sum\limits_{q=0}^{p} \phi_i^q = u_i^p$. Then, we prove that u_i^p converge to u_i. For all $p \geq 1$, by using (H_1), 5

$$\begin{aligned} |\mathcal{K}(x_i, x_i, u_i^p) - \mathcal{K}(x_i, x_i, u_i^{p-1})| &\leq h|\omega_i| \, |\frac{\partial \mathcal{K}}{\partial x}(x_i, x_i, u_i^{p-1)}) - \frac{\partial \mathcal{K}}{\partial x}(x_i, x_i, u_i^{p-2})|, \\ &\leq LhW \, |u_i^{p-1} - u_i^{p-2}|. \end{aligned} \tag{4.7}$$

On the other hand, under the assumption $(\mathcal{H})$ (7), we have

$$|\mathcal{K}(x_i, x_i, u_i^p) - \mathcal{K}(x_i, x_i, u_i^{p-1})| \geq \theta |u_i^p - u_i^{p-1}|. \tag{4.8}$$

According to equalities (4.7) and (4.8), we obtain

$$|u_i^p - u_i^{p-1}| \leq \frac{LhW}{\theta} |u_i^{p-1} - u_i^{p-2}|, \tag{4.9}$$

which gives

$$|\phi_i^p| \leq \frac{LhW}{\theta} |\phi_i^{p-1}|. \tag{4.10}$$

By induction, we can prove that

$$|\phi_i^p| \leq \left(\frac{LhW}{\theta}\right)^p |\phi_{i-1}^0|. \tag{4.11}$$

Finally, we get

$$\sum_{q=0}^{p} |\phi_i^q| = \sum_{q=0}^{p} \left(\frac{LhW}{\theta}\right)^q |u_i^0|.$$

Assuming that h is small enough so that $\dfrac{LhW}{\theta} < 1$, we can show that $\sum\limits_{q\geq 1} \left(\dfrac{LhW}{\theta}\right)^q$ is convergent. Therefore, $\sum\limits_{q=0}^{p} \phi_i^q$ is convergent. So, $\lim\limits_{p\to+\infty} u_i^p = u_i$. It remains to check whether this limit satisfies the required our equation.
Recalling (4.5)

$$\mathcal{K}(x_i, x_i, u_i^{p+1}) = S_i - hw_i \frac{\partial \mathcal{K}}{\partial x}(x_i, x_i, u_i^p), \quad p \geq 1.$$

Then,

$$\lim_{p\to+\infty} \mathcal{K}(x_i, x_i, u_i^{p+1}) = S_i - hw_i \lim_{p\to+\infty} \frac{\partial \mathcal{K}}{\partial x}(x_i, x_i, u_i^p),$$

since, $\mathcal{K}$ and $\dfrac{\partial \mathcal{K}}{\partial x} \in C^0([a,b]^2 \times \mathbb{R})$. This leads

$$\mathcal{K}(x_i, x_i, u_i) = S_i - hw_i \frac{\partial \mathcal{K}}{\partial x}(x_i, x_i, u_i),$$

which gives the result.
It remains to show the existence of the solution. Let $\{u_i\}_{0\le i\le n}$ and $\{v_i\}_{0\le i\le n}$ be solutions of system (4.7) and (4.8) respectively. Then, for all $i \ge 1$ fixed

$$|\mathcal{K}(x_i, x_i, u_i) - \mathcal{K}(x_i, x_i, v)| \ge \theta |u_i - v_i|. \tag{4.12}$$

On the other hand, we have

$$\begin{aligned} |\mathcal{K}(x_i, x_i, u_i) - \mathcal{K}(x_i, x_i, v)| &\le hw \left| \frac{\partial \mathcal{K}}{\partial}(x_i, x_i, u_i) - \frac{\partial \mathcal{K}}{\partial}(x_i, x_i, v_i) \right|, \\ &\le hWL|u_i - v_i|. \end{aligned} \tag{4.13}$$

Next, from (4.12) and (4.13) it follows that

$$|u_i - v_i| \le \frac{hWL}{\theta} |u_i - v_i|. \tag{4.14}$$

Since $\dfrac{hWL}{\theta} < 1$, we get $u_i = v_i$. This proves the uniqueness of the solution of the system (4.4) and completes the proof of Theorem 4.1. □

4.1.1. Approximate solution convergence of discrete problem. First, we define the local consistency error as

$$\delta_n(h, x_i) = \int_0^{x_i} \mathcal{K}(x_i, t, u(t))\, dt - h \sum_{j=0}^{i} \omega_j \frac{\partial \mathcal{K}}{\partial x}(x_i, x_j, u(x_j)).$$

We say the numerical method is consistent if

$$\lim_{h\to 0} \left(\max_{0\le i\le n} |\delta_n(h, x_i)| \right) = 0.$$

Now, we establish the convergence of the approximate solution u_i.

Theorem 4.2. *If the approximation method is consistent, then*

$$\lim_{h\to 0} \left(\max_{0\le i\le n} |u_i - u(x_i)| \right) = 0.$$

Proof. For n large enough and $i = 0, 1, \dots n$, we have

$$\begin{aligned} |\mathcal{K}(x_i, x_i, u_i) - \mathcal{K}(x_i, x_i, u(x_i))| &\le h \sum_{j=0}^{i} |\omega_j| \left| \frac{\partial \mathcal{K}}{\partial x}(x_i, x_j, u_j) - \frac{\partial \mathcal{K}}{\partial x}(x_i, x_j, u(x_j)) \right| + |\delta_n(h, x_i)|, \\ &\le Lh|\omega_i||u_i - u(x_i)| + Lh \sum_{j=0}^{i-1} |\omega_j|\, |u_j - u(x_j)| + |\delta_n(h, x_i)|. \end{aligned}$$

But, using $H)_1, 7$ we get

$$\theta\, |u_i - u(x_i)| \le |\mathcal{K}(x_i, x_i, u_i) - \mathcal{K}(x_i, x_i, u(x_i))|.$$

Then, we obtain

$$\theta|u_i - u(x_i)| \leq Lh|\omega_i|\,|u_i - u(x_i)| + Lh\sum_{j=0}^{i-1}|\omega_j|\,|u_j - u(x_j)| + |\delta_n(h,x_i)|.$$

Thus

$$\theta|u_i - u(x_i)| \leq LhW\,|u_i - u(x_i)| + LhW\sum_{j=0}^{i-1}|u_j - u(x_j)| + |\delta_n(h,x_i)|,$$

so that

$$|u_i - u(x_i)| \leq \frac{LhW}{\theta - LhW}\sum_{j=0}^{i-1}|u_j - u(x_j)| + \frac{\max\limits_{0\leq i\leq n}|\delta_n(h,x_i)|}{\theta - LhW}.$$

Applying Gronwel's lemma [16], we obtain

$$|u_i - u(x_i)| \leq \frac{1}{\theta - LhW}\left(1 + \frac{LhW}{\theta - LhW}\right)^{i-1}\left(\max_{0\leq i\leq n}|\delta_n(h,x_i)| + LhW|u_0 - u(x_0)|\right).$$

Moreover, we have

$$\left(1 + \frac{LhW}{\theta - LhW}\right)^{i-1} \leq \left(1 + \frac{LhW}{\theta - LhW}\right)^{n},$$

and

$$\lim_{n\to\infty}\left(1 + \frac{LhW}{\theta - LhW}\right)^{n} < +\infty.$$

This means that there is a constant $\mu > 0$ such that

$$\forall n \geq 1,\ \max_{0\leq i\leq n}\frac{1}{\theta - LhW}\left(1 + \frac{LhW}{\theta - LhW}\right)^{i-1} \leq \mu.$$

This implies that

$$\max_{0\leq i\leq n}|u_i - u(x_i)| \leq \mu\left(\max_{0\leq i\leq n}|\delta_n(h,x_i)| + LhW|u_0 - u(x_0)|\right),$$

and when h converges to 0, we get convergence of n to ∞. The desired result follows. □

4.2. Linearization

We start by presenting the process of Newton's method. Let us recalling that en general, Newton's method is applied to solve many problems of applied mathematics which have the general form:

$$\begin{cases} \text{Find the solution } \ v \in \mathbb{R}\ of \\ \qquad g(v) = 0, \end{cases} \tag{4.15}$$

where g is a non linear function. For $i \geq 1$ fixed, we define the non-linear functional ψ_i as:

$$\begin{aligned} \psi_i:\ \ \mathbb{R} &\longrightarrow \mathbb{R} \\ v &\longmapsto \psi_i(v) = \mathcal{K}(x_i,x_i,v) + hw_i\frac{\partial\mathcal{K}}{\partial x}(x_i,x_i,v) - S_i. \end{aligned} \tag{4.16}$$

Assume that $\dfrac{\partial\mathcal{K}}{\partial x}$ verifies the following hypotheses

$$(H_2)\left\|\ \frac{\partial^2\mathcal{K}}{\partial v\partial x} \in C^0([0,X]^2\times\mathbb{R},\mathbb{R})\right.$$

Then, we get $\psi_i \in C^2(\mathbb{R}, \mathbb{R})$.
The system to solve has the following form:

$$\begin{cases} \text{Find the solution } \ u_i \in \mathbb{R} \ of \\ \\ \qquad \psi_i(u_i) = 0. \end{cases} \tag{4.17}$$

Applying Newton's method, we obtain the following system:

$$\begin{cases} \qquad u_i^0 \in \mathbb{R}, \\ \\ u_i^{k+1} = u_i^k - \dfrac{\psi_i(u_i^k)}{\psi_i'(u_i^k)}. \end{cases} \tag{4.18}$$

Theorem 4.3. *For i fixed, let ψ_i be a non-linear function defined by (4.16) of class C^2. Let $R > 0$, $J_{i,R}$ be closed ball such that $J_{i,R} = [u_i - R, u_i + R] \subseteq I_i$ and $C = \dfrac{\max|\psi_i''(v)|}{2\min|\psi_i'(v)|}$. If $CR < 1$ and if the starting approximation u_i^0 in $J_{i,R}$, then for all u_i^k in $J_{i,R}$ and*

$$|u_i^{k+1} - u_i| \leq C|u_i^k - u_i|^2,$$

where

$$C|u_i^k - u_i|^2 \leq \Big(C|u_i^0 - u_i|\Big)^{2k} \leq (CR)^{2k}.$$

Proof. See [37] and [38]. □

Theorem 4.4. *Let $u(x_i)$ be a solution of (4.2) and u_i^k is an approxiamate solution of the the iterative system (4.18), then*

$$\lim_{n\to+\infty}\lim_{k\to+\infty}\left(\max_{0\leq i\leq n}|u(x_i) - u_i^k|\right) = 0.$$

Proof. For n and k large enough, we have

$$\max_{0\leq i\leq n}|u(x_i) - u_i^k| \leq \max_{0\leq i\leq n}|u(x_i) - u_i| + \max_{0\leq i\leq n}|u_i - u_i^k|.$$

Theorem 4.2 gives

$$\max_{0\leq i\leq n}|u_i - u(x_i)| \leq \mu\left(\max_{0\leq i\leq n}|\delta_n(h, x_i)| + LhW|u_0 - u(x_0)|\right),$$

and from Theorem 4.3 we get

$$\max_{0\leq i\leq n}|u_i - u_i^k| \leq \frac{(CR)^{2k}}{C}.$$

Then,

$$\max_{0\leq i\leq n}|u(x_i) - u_i^k| \leq \mu\left(\max_{0\leq i\leq n}|\delta_n(h, x_i)| + LhW|u_0 - u(x_0)|\right) + \frac{(CR)^{2k}}{C}.$$

So, when $n \to +\infty$ and $k \to +\infty$, we get the desired result. □

5. Numerical Examples

In this section, we give two numerical examples to illustrate the efficiency and accuracy of proposed method. In the following examples, we calculate u_i according the scheme (4.18), and we define the discrete error as:

$$err_n = \max_{0\leq i\leq n} |u(x_i) - u_i^k|.$$

Let us start with the equation:

$$\forall x \in [0,1], \int_0^x \frac{1}{x+t+9+\exp(u(t))}\, dx = \frac{1}{2}\log\left(\frac{3x+10}{x+10}\right), \tag{5.1}$$

where the exact solution is $u(x) = \log(x+1)$. Table 1 shows the error magnitudes between the approximate and exact solutions. The results we present in the table were obtained using the MATLAB program.

n \ k	10	20	30	50				
10	9.6047e-04	7.799171	9.6047e-04	10.994035	9.6047e-04	14.677863	9.6047e-04	25.032011
50	2.0446e-04	27.800183	2.0446e-04	69.370806	2.0446e-04	71.267143	2.0446e-04	273.740221
100	1.0290e-04	48.720499	1.0290e-04	91.716690	1.0290e-04	158.136187	1.0290e-04	5526.485901
500	7.9802e-04	225.026614	1.0705e-04	429.065173	1.8603e-05	2.0682e-05	7.3098e-06	3082.171232
1000	4.0433e-04	840.972380	5.3099e-05	1400.235219	8.2444e-06	1562.552823	1.8275e-06	4724.151322

Table 1: Errors of exact and approximate solutions.

In the table 1, in each row, we choose n and then change the value of k. We notice that when n is equal to 10 and 50, the error does not change. But, after having chosen a large n, the more the number of iterations k increases, the less the error becomes. So the approximate solution converges to the exact solution. In the next figure, we plot the approximate and exact solutions to observe the difference between them.

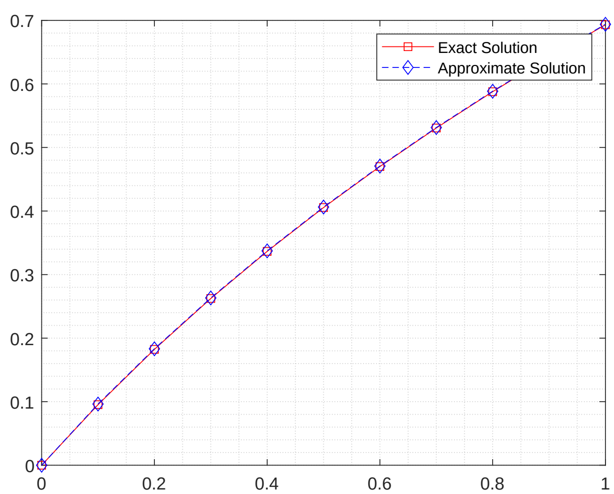

Figure 1: Approximate and exact solutions with $h = 0.1$ of equation (5.1).

We give another equation

$$\forall x \in [0,3], \int_0^x \frac{\exp(x)t^2+1}{\cos(t)^2+1+u(t)^2}\,dt = \frac{x}{2}+\frac{x^3\exp(x)}{6}, \tag{5.2}$$

and the exact solution $u(x)=\sin(x)$. Let introduce this table, which explain the error between the numerical and exact solution in all points x_i. What we are going to present in the table are calculated using MATLAB.

n \ k	10	time(s)	20	time(s)	30	time(s)	50	time(s)
10	0.0260	11.175920	0.0176	22.756830	0.0176	26.460829	0.0176	37.429666
50	0.0068	38.840905	0.0013	87.966106	7.3009e-04	109.458969	7.3009e-04	158.506082
100	0.0037	59.144494	5.7459e-04	119.415460	1.8268e-04	174.320379	1.8268e-04	231.792491
500	7.9802e-04	249.483759	1.0705e-04	597.724081	1.8603e-05	5109.798293	7.3098e-06	1204.280340
1000	4.0433e-04	1003.764503	5.3099e-05	3112.587714	8.2444e-06	23388.40	1.8275e-06	2560.750899

Table 2: Errors of exact and approximate solutions.

In each row of the table 2, we will set n to a certain value and then change to k until we get the best possible error. From one row to the next, we increase the value of n, we notice that the error decreases each time. Our goal through the two tables 1 and 2 is to specify everywhere the number of divisions n of the interval $[a,b]$ the larger. We obtain an error close to zero. Thus, we guarantee the convergence of the numerical solution to the exact solution.

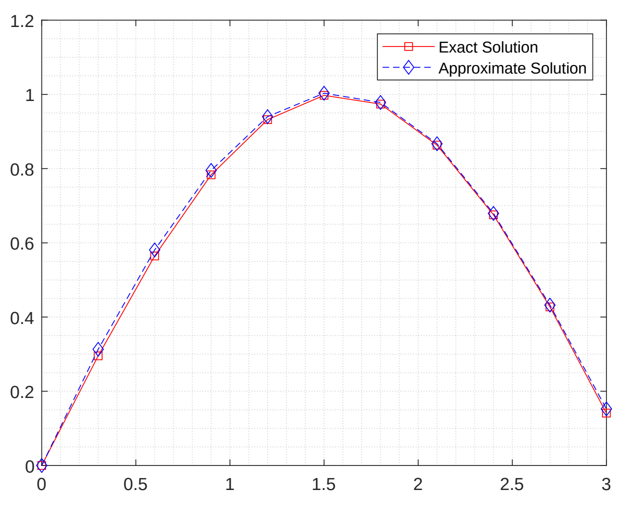

Figure 2: Approximate and exact solution with $h=0.1$ of equation (5.2).

6. Conclusion

In this paper, we are interested in the analytical and numerical study of non-linear Volterra integral equation of the first kind. A sufficient condition has been introduced in order to prove the existence and uniqueness of the analytical solution. Then, for the numerical approach, we started with the discretization of our problem by applying the principle of Nyström's method. Next, we obtained at each iteration i a non-linear equation to solve. For this reason, we applied the principle of Newton's method to reformulate the discrete nonlinear equation to a linear equation. We have presented two convergence theorems: the first one for the convergence of the discrete solution u_i to $u(x_i)$ and the second one for the convergence of the iterative solution u_i^k to the discrete solution u_i. At the end, we have given two numerical examples to illustrate the efficiency of the proposed numerical method.

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