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On Stability of Second Order Pantograph Fractional Differential Equations in Weighted Banach Space

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Abstract: This work investigates a weighted Banach space second order pantograph fractional differential equation. The considered equation is of second order, expressed in terms of the Caputo–Hadamard fractional operator, and constructed in a general manner to accommodate many specific situations. The asymptotic stability of the main equation's trivial solution has been given. The primary theorem was demonstrated in a unique manner by employing the Krasnoselskii's fixed point theorem. We provide a concrete example that supports the theoretical findings.

Keywords: Caputo–Hadamard fractional derivative; pantograph fractional differential equations; Krasnoselskii's fixed point theorem; asymptotic stability

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1. Introduction

Fractional calculus (FC) is a branch of mathematical analysis that investigates alternative approaches to define the real or complex number powers of the differentiation and integration operators. FC has numerous and varied applications in the domains of engineering and science, including optics, signals processing, viscoelasticity, fluid mechanics, electrochemistry, biological population models, and electromagnetics; consult for instance [1,2].

Fractional differential equations (FDE) are often treated via many types of fractional derivatives, including the two most well-known, Riemann–Louville fractional derivative (RLFD) and Caputo fractional derivative (CFD), which have drawn the majority of researchers' attention. On the other hand, the Caputo–Hadamard fractional derivative (CHFD) is advantageous since it uses a logarithmic kernel with memory, which is appropriate for describing complicated systems. The works in [3,4] provide useful summaries and applications of how and where the CHFD arises. FDE within CHFD are a very important class of equations, and their applications can be found in a variety of engineering and scientific disciplines, such as mechanics, biology, chemistry, physics, the stability and instability of geodesics on Riemannian varieties, Hamiltonian systems, and technical engineering sciences [5,6].

A pantograph is a mechanical linkage that is used to copy and scale a drawing or image. It consists of a series of interconnected parallelograms that are able to maintain the same shape, even as they change size. The pantograph was first invented in the 17th century and has since been used for a wide range of applications. The pantograph can be modeled mathematically using a set of equations known as the pantograph equations. These equations describe the relationships between the various lengths and angles of the parallelograms that make up the pantograph. Important application of the pantograph is in the field of engineering including the design of machines and mechanisms that require precise scaling and copying of movements. For example, a pantograph can be used to design a linkage system for a robot arm that needs to move in a specific way. There are some other pertinent applications of pantograph in electrodynamics [7], number theory [8], and the energy absorbed by an electronic locomotive [9–12].

Due to its importance, numerous researchers have recently focused on the fractional pantograph equation and made helpful contributions in this direction. In [13], the authors studied the existence of solutions of nonlinear fractional pantograph equations. Existence and uniqueness results for nonlinear neutral pantograph equations with generalized fractional derivative were proved in [14]. Several significant conclusions have been made on this subject using different iterations of the pantograph equation and various types of fractional operators [15–20]. Fixed point theorems are frequently used to demonstrate the solutions' existence and uniqueness. Few results used fixed point methods to show the stability of the solutions; for example in the recent paper [21], stability results in the sense of Ulam–Hyers and its generalized form of stability are established for generalized hybrid discrete pantograph equation. The fact that all of the aforementioned research was done using first-order fractional operators is noteworthy. The literature rarely considers works on second-order fractional pantograph equations [22].

In [23], Agarwal et al. used the Krasnoselskii's fixed point theorem (KFPT) to prove the existence of solutions for the following neutral FDE with bounded delay

$$\begin{cases} {}^{C}D^{\alpha}(X(t) - g(t, X_t)) = f(t, X_t),\ t \geq t_0, \\ X_{t_0} = \chi, \end{cases}$$

where ${}^{C}D^{\alpha}$ is CFD of order $0 < \alpha < 1$, $f, g : [t_0, \infty) \times C([-r, 0], \mathbb{R}^n) \to \mathbb{R}^n$ and $\chi \in C([-r, 0], \mathbb{R}^n)$. Further, in [24], Ge and Kou employed the KFPT in weighted Banach space to discuss the asymptotic stability (AS) of the zero solution for the following nonlinear FDE:

$$\begin{cases} {}^{C}D^{\alpha}_{0+}X(t) = f(t, X(t)),\ t \geq 0, \\ X(0) = X_0,\ X'(0) = X_1, \end{cases}$$

where $1 < \alpha < 2$, $X_0, X_1 \in \mathbb{R}$, $f : \mathbb{R}^{+} \times \mathbb{R} \to \mathbb{R}$ is a continuous function where $f(t, 0) \equiv 0$, and $\mathbb{R}^{+} = [0, \infty)$. Motivated by the aforementioned arguments, the pertinent literature that is currently available [25–31], and the fact that the second order pantograph fractional differential equations are not frequently used in the literature, we consider the following second order pantograph CHFD equation:

$$\begin{cases} \mathfrak{D}^{\alpha}_{1}\varphi(\varsigma) - \mathfrak{D}^{\alpha-1}_{1}\mathcal{H}(\varsigma, \varphi(1+\vartheta\varsigma)) = \mathcal{F}(\varsigma, \varphi(\varsigma), \varphi(1+\vartheta\varsigma)),\ \varsigma \geq 1, \\ \varphi(1) = \varphi_0,\ \varphi'(1) = \varphi_1, \end{cases} \tag{1}$$

where $\vartheta \in (0, 1)$, $1 < \alpha \leq 2$, $\varphi_0, \varphi_1 \in \mathbb{R}$, $\mathfrak{D}^{\alpha}_{1}$ is the standard CHFD, $\mathcal{H} : [1, \infty) \times \mathbb{R} \to \mathbb{R}$ and $\mathcal{F} : [1, \infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions (CFs) and the nonlinearities satisfy $\mathcal{H}(\varsigma, 0) = \mathcal{F}(\varsigma, 0, 0) = 0$. The solution of Equation (1) is denoted by $\varphi(\varsigma)$. The objective of the paper is to prove the AS of the zero solution of Equation (1). Our method relies on KFPT, which calls for the conversion of Equation (1) into an integral equation combining two mappings, one of which is a compact mapping and the other is a contraction mapping. The main contributions of this paper are summarized by

(i) Equation (1) is a second order pantograph fractional differential equation that has received less attention in the literature.
(ii) Rather than being used to demonstrate the system's solvability, the KFPT is utilized to demonstrate the AS of the primary equation.
(iii) Equation (1) is exposed in generic form and encompasses a wide range of particular cases.

The following is the structure of this article. Section 2 discusses certain notations and lemmas, as well as some preliminary results that will be needed in following section. Section 3 is devoted to the main results, where we prove the stability of Equation (1). The application, which includes an example, concludes the paper in Section 4.

2. Fundamental Concepts

This section introduces some basic definitions, essential lemmas and fundamental theorems that are utilized throughout this work. For more details, see [1,2,32–40].

Definition 1 ([1,2])**.** *The Riemann–Liouville fractional integral (RLFI) of order $\alpha > 0$ for a function $\varphi : [0, +\infty) \to \mathbb{R}$ is defined as*

$$I^{\alpha}\varphi(\varsigma) = \frac{1}{\Gamma(\alpha)} \int_0^{\varsigma} (\varsigma - \tau)^{\alpha-1} \varphi(\tau) d\tau,$$

where

$$\Gamma(\alpha) = \int_0^{\infty} e^{-\varsigma} \varsigma^{\alpha-1} d\varsigma,$$

is the Euler gamma function.

Definition 2 ([1,2])**.** *The Hadamard fractional integral of order $\alpha > 0$ for a CF $\varphi : [1, +\infty) \to \mathbb{R}$ is defined as*

$$\mathfrak{I}_1^{\alpha}\varphi(\varsigma) = \frac{1}{\Gamma(\alpha)} \int_1^{\varsigma} \left(\log \frac{\varsigma}{\tau}\right)^{\alpha-1} \varphi(\tau) \frac{d\tau}{\tau}.$$

Definition 3 ([1,2])**.** *The RLFD of order $\alpha > 0$ for a CF $\varphi : [0, +\infty) \to \mathbb{R}$ is defined as*

$$D^{\alpha}\varphi(\varsigma) = \frac{1}{\Gamma(n - \alpha)} \int_0^{\varsigma} (\varsigma - \tau)^{n-\alpha-1} \varphi^{(n)}(\tau) d\tau, \ n - 1 < \alpha < n, \ n \in \mathbb{N}.$$

Definition 4 ([1,2])**.** *The CHFD of order $\alpha > 0$ for a CF $\varphi : [1, +\infty) \to \mathbb{R}$ is defined as*

$$\mathfrak{D}_1^{\alpha}\varphi(\varsigma) = \frac{1}{\Gamma(n - \alpha)} \int_1^{\varsigma} \left(\log \frac{\varsigma}{\tau}\right)^{n-\alpha-1} \delta^n \varphi(\tau) \frac{d\tau}{\tau}, \ n - 1 < \alpha < n,$$

where $\delta^n = \left(\varsigma \frac{d}{d\varsigma}\right)^n, n \in \mathbb{N}$.

Lemma 1 ([1,2])**.** *For $\alpha \in (n-1, n]$, $n \in \mathbb{N}$. The equality $\left(\mathfrak{I}_1^{\alpha}\mathfrak{D}_1^{\alpha}\varphi\right)(\varsigma) = 0$ is true if and only if*

$$\varphi(\varsigma) = \sum_{k=1}^{n} c_k (\log \varsigma)^{\alpha-k} \ \text{for} \ \varsigma \in [1, \infty),$$

where $c_k \in \mathbb{R}$ is arbitrary constant for $k = 1, \ldots, n$.

Lemma 2 ([1,2])**.** *Let $m - 1 < \alpha \leq m$, $m \in \mathbb{N}$ and $\varphi \in C^{n-1}[1, \infty)$. So*

$$\mathfrak{I}_1^{\alpha}[\mathfrak{D}_1^{\alpha}\varphi(\varsigma)] = \varphi(\varsigma) - \sum_{k=0}^{m-1} \frac{\left(\delta^k \varphi\right)(1)}{\Gamma(k+1)} (\log \varsigma)^k.$$

Lemma 3 ([1,2]). *For all $\rho > 0$ and $\nu > -1$,*

$$\frac{1}{\Gamma(\rho)}\int_1^{\varsigma}\left(\log\frac{\varsigma}{\tau}\right)^{\rho-1}(\log\tau)^{\nu}\frac{d\tau}{\tau}=\frac{\Gamma(\nu+1)}{\Gamma(\rho+\nu+1)}(\log\varsigma)^{\rho+\nu}.$$

Lemma 4 ([1,2]). *Let $\varphi(\varsigma)=(\log\varsigma)^{\rho}$, where $\rho\geq 0$ and $m-1<\alpha\leq m$, $m\in\mathbb{N}$. Then,*

$$\mathfrak{D}_1^{\alpha}\varphi(\varsigma)=\begin{cases}0 & \text{if } \rho\in\{0,1,\ldots,m-1\},\\ \frac{\Gamma(\nu+1)}{\Gamma(\rho+\nu+1)}(\log\varsigma)^{\rho-\nu} & \text{if } \rho\in\mathbb{N}, \rho\geq m \text{ or } \rho\notin\mathbb{N}, \rho>m-1.\end{cases}$$

Remark 1. *By Definitions 2 and 4 and Lemma 2, we have*

(i) *Let $\Re(\alpha)>0$. If φ is continuous on $[1,+\infty)$, then $\mathfrak{D}_1^{\alpha}\mathfrak{I}_1^{\alpha}\varphi(\varsigma)=\varphi(\varsigma)$ holds for all $[1,+\infty)$.*

(ii) *Let $\varphi(\varsigma)=c$. Then, $\mathfrak{D}_1^{\alpha}\varphi(\varsigma)=0$.*

Define the Banach space

$$\mathcal{E}=\left\{\varphi\in C([1,+\infty)):\sup_{\varsigma\geq 1}|\varphi(\varsigma)|/\mathbb{G}(\log\varsigma)<\infty\right\},$$

which serves as a crucial link in our research. Let $\mathbb{G}:[0,+\infty)\to[1,+\infty)$ be a strictly increasing CF with $\mathbb{G}(0)=1$, $\mathbb{G}(\log\varsigma)\to\infty$ as $\varsigma\to\infty$,

$$\mathbb{G}(\log\tau)\mathbb{G}(\log\frac{\varsigma}{\tau})\leq\mathbb{G}(\log\varsigma)\text{ for all }1\leq\tau\leq\varsigma\leq\infty.$$

Then, $\mathcal{E}$ is a Banach space equipped by $\|\varphi\|=\sup_{\varsigma\geq 1}\frac{|\varphi(\varsigma)|}{\mathbb{G}(\log\varsigma)}$. For more information about this space, one may consult [35]. Define

$$\|\Psi\|_{\varsigma}=\max\{|\Psi(\tau)|:1\leq\tau\leq\varsigma\},$$

for any $\varsigma\geq 1$, where $\Psi\in C([1,+\infty))$. Then, we obtain $\mathfrak{B}(\varepsilon)=\{\varphi\in\mathcal{E}:\|\varphi(\varsigma)\|\leq\varepsilon$ for $\varsigma\in[1,+\infty)\}$ for any $\varepsilon>0$.

Lemma 5. *Let $\mathcal{W}(\varsigma)\in C([1,+\infty))$. Then, $\varphi\in C([1,+\infty))$ is a solution of the Cauchy type problem*

$$\begin{cases}\mathfrak{D}_1^{\alpha}\varphi(\varsigma)=\mathcal{W}(\varsigma),\ \varsigma\geq 1,\ 1<\alpha<2,\\ \varphi(1)=\varphi_0,\ \varphi'(1)=\varphi_1,\end{cases}\tag{2}$$

if and only if f φ is a solution of

$$\begin{cases}\varsigma\varphi'(\varsigma)=\mathfrak{I}_1^{\alpha-1}\mathcal{W}(\varsigma)+\varphi_1,\ \varsigma\geq 1,\\ \varphi(1)=\varphi_0.\end{cases}\tag{3}$$

Proof. First, we suggest that for any $0<\gamma<1$, if $\psi\in C[1,+\infty)$, then $\left(\mathfrak{I}_1^{\gamma}\psi\right)(1)=0$. In fact, since

$$\mathfrak{I}_1^{\gamma}\psi(\varsigma)=\frac{1}{\Gamma(\gamma)}\int_1^{\varsigma}(\log\frac{\varsigma}{\tau})^{\gamma-1}\psi(\tau)\frac{d\tau}{\tau},$$

we can conclude that

$$\left|\mathfrak{I}_1^{\gamma}\psi(\varsigma)\right|=\frac{1}{\Gamma(\gamma)}\left|\int_1^{\varsigma}(\log\frac{\varsigma}{\tau})^{\gamma-1}\psi(\tau)\frac{d\tau}{\tau}\right|\leq\frac{\|\psi\|_{\varsigma}}{\Gamma(\gamma+1)}(\log\varsigma)^{\gamma}\to 0\text{ as }\varsigma\to 1.$$

Let $\varphi\in C[1,+\infty)$ be a solution of Equation (2). For any $\varsigma\geq 1$, Definition 4 shows that

$$\mathfrak{D}_1^{\alpha}\varphi(\varsigma)=\left(\mathfrak{D}_1^{\alpha-1}\mathfrak{D}_1^{1}\varphi\right)(\varsigma)=\mathcal{W}(\varsigma).$$

Due to Lemma 2, we have

$$\varsigma\varphi'(\varsigma) = \varphi'(1) + \mathfrak{I}_1^{\alpha-1}\mathcal{W}(\varsigma) = \mathfrak{I}_1^{\alpha-1}\mathcal{W}(\varsigma) + \varphi_1.$$

This indicates that $\varphi(\varsigma)$ is a solution of Equation (3). On other hand, assume that $\varphi(\varsigma)$ is a solution of Equation (3). For all $\varsigma \geq 1$, one can straightforwardly see

$$\mathfrak{D}_1^{\alpha}\varphi(\varsigma) = \mathfrak{D}_1^{\alpha-1}(\varsigma\varphi'(\varsigma)) = \left(\mathfrak{D}_1^{\alpha-1}\mathfrak{I}_1^{\alpha-1}\mathcal{W}\right)(\varsigma) + \mathfrak{D}_1^{\alpha-1}\varphi_1 = \mathcal{W}(\varsigma).$$

Further, we note that $\mathcal{W}(\varsigma) \in C[1,+\infty)$, and thus we have $\varphi'(1) = \mathfrak{I}_1^{\alpha-1}\mathcal{W}(1) + \varphi_1 = \varphi_1$. □

Lemma 6. *Let $k \in \mathbb{R}$. Then, $\varphi \in C([1,+\infty))$ is a solution of (1) if and only if*

$$\begin{aligned}\varphi(\varsigma) &= \varphi_0 e^{-k\varsigma} + (\varphi_1 - \mathcal{H}(1,\varphi(1+\vartheta)))\int_1^{\varsigma} e^{-k(\varsigma-\tau)}\frac{d\tau}{\tau} \\ &\quad + \int_1^{\varsigma} e^{-k(\varsigma-\tau)}(k\tau\varphi(\tau) + \mathcal{H}(\tau,\varphi(1+\vartheta\tau)))\frac{d\tau}{\tau} \\ &\quad + \frac{1}{\Gamma(\alpha-1)}\int_1^{\varsigma}\int_v^{\varsigma} e^{-k(\varsigma-\tau)}\left(\log\frac{\varsigma}{v}\right)^{\alpha-2}\frac{d\tau}{\tau}\mathcal{F}(v,\varphi(v),\varphi(1+\vartheta v))\frac{dv}{v}. \end{aligned} \tag{4}$$

Proof. Let $\varphi \in C([1,+\infty))$ be a solution of Equation (1). By Lemma 5, obtain

$$\begin{cases} \varsigma\varphi'(\varsigma) = \mathfrak{I}_1^{\alpha-1}\left[\mathcal{F}(\varsigma,\varphi(\varsigma),\varphi(1+\vartheta\varsigma)) + \mathfrak{D}_1^{\alpha-1}\mathcal{H}(\varsigma,\varphi(1+\vartheta\varsigma))\right] + \varphi_1,\ \varsigma \geq 1, \\ \varphi(1) = \varphi_0. \end{cases}$$

It follows that

$$\begin{cases} \varsigma\varphi'(\varsigma) = \frac{1}{\Gamma(\alpha-1)}\int_1^{\varsigma}(\log\frac{\varsigma}{\tau})^{\alpha-2}\mathcal{F}(\tau,\varphi(\tau),\varphi(1+\vartheta\tau))\frac{d\tau}{\tau} \\ \qquad +\mathcal{H}(\varsigma,\varphi(1+\vartheta\varsigma)) - \mathcal{H}(1,\varphi(1+\vartheta)) + \varphi_1,\ \varsigma \geq 1, \\ \varphi(1) = \varphi_0. \end{cases} \tag{5}$$

Rewrite Equation (5) as

$$\begin{cases} \varphi'(\varsigma) + k\varphi(\varsigma) = k\varphi(\varsigma) + \frac{1}{\varsigma}\frac{1}{\Gamma(\alpha-1)}\int_1^{\varsigma}(\log\frac{\varsigma}{\tau})^{\alpha-2}\mathcal{F}(\tau,\varphi(\tau),\varphi(1+\vartheta\tau))\frac{d\tau}{\tau} \\ \qquad +\frac{1}{\varsigma}\mathcal{H}(\varsigma,\varphi(1+\vartheta\varsigma)) - \frac{1}{\varsigma}\mathcal{H}(1,\varphi(1+\vartheta)) + \frac{1}{\varsigma}\varphi_1,\ \varsigma \geq 1, \\ \varphi(1) = \varphi_0. \end{cases}$$

By the help of the formula of variation of parameters, we end up with Equation (4). The reverse can be easily justified and hence is omitted. The proof is complete. □

Definition 5. *The trivial solution $\varphi = 0$ of Equation (1) is*

(i) *Stable in $\mathcal{E}$, if for every $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ as such $|\varphi_0| + |\varphi_1| \leq \delta$ requires that the solution $\varphi(\varsigma) = \varphi(\varsigma,\varphi_0,\varphi_1)$ exists for all $\varsigma \geq 1$ and satisfies $\|\varphi\| \leq \varepsilon$.*

(ii) *Asymptotically stable, if (i) investigator and there is a number $\sigma > 0$ such that $|\varphi_0| + |\varphi_1| \leq \sigma$ implies $\lim_{\varsigma\to\infty}\|\varphi(\varsigma)\| = 0$.*

The KFPT will be presented to demonstrate the AS of the zero solution of Equation (1).

Theorem 1 (Krasnoselskii [40]). *If Θ is a non-empty closed convex subset of a Banach space $(Z,\|.\|)$, P and Q map Θ into Z, then*

(i) *$P\varphi + Q\mathfrak{y} \in \Theta$ for all $\varphi,\mathfrak{y} \in \Theta$,*

(ii) *P is continuous and $P\Theta$ is contained in a compact set of Z,*

(iii) *Q is a contraction with $l < 1$.*

Thus, there is a $\varphi \in \Theta$ *with* $P\varphi + Q\varphi = \varphi$.

To complete proving the main results, we need the following theorem.

Theorem 2 ([35]). *Let* $\mathcal{E}$ *be a Banach space and* $\aleph \subset \mathcal{E}$. *Thus,* $\aleph$ *is relatively compact in* $\mathcal{E}$ *if the next assumptions are fulfilled:*

(i) $\{\varphi(\varsigma)/\mathbb{G}(\log\varsigma) : \varphi \in \aleph\}$ *is uniformly bounded,*
(ii) *On any compact interval of* $[1, +\infty)$, *the set* $\{\varphi(\varsigma)/\mathbb{G}(\log\varsigma) : \varphi \in \aleph\}$ *is equicontinuous,*
(iii) $\{\varphi(\varsigma)/\mathbb{G}(\log\varsigma) : \varphi \in \aleph\}$ *is equiconvergent for any given* $\varepsilon > 0$, *there exists a* $T_0 > 1$ *as such for all* $\varphi \in \aleph$ *and* $\varsigma_1, \varsigma_2 > T_0$, *if holds*

$$|\varphi(\varsigma_2)/\mathbb{G}(\log\varsigma_2) - \varphi(\varsigma_1)/\mathbb{G}(\log\varsigma_1)| < \varepsilon.$$

3. Main Results

Before mentioning and proving the main results, we set forth some essential conditions.

$(\Sigma 1)$ $\mathcal{H}$ and $\mathcal{F}$ are CFs and $\mathcal{H}(\varsigma, 0) = \mathcal{F}(\varsigma, 0, 0) = 0$. Assuming $\mathcal{H}$ is locally Lipschitz continuous in φ. This means there is a $L_{\mathcal{H}} > 0$ as that if $|\varphi|, |\mathfrak{y}| \leq l$ then

$$|\mathcal{H}(\varsigma, \varphi) - \mathcal{H}(\varsigma, \mathfrak{y})| \leq L_{\mathcal{H}}\|\varphi - \mathfrak{y}\|.$$

$(\Sigma 2)$ There exists a $\beta_1 \in (0, 1)$ as such

$$\beta_1\left(1 + \frac{L_{\mathcal{H}}}{|k|}\right) < 1, \tag{6}$$

and

$$e^{-k\varsigma}/\mathbb{G}(\log\varsigma) \in BC([1, +\infty)) \cap L^1([1, +\infty)),\ |k|\int_1^{\varsigma} e^{-kv}/\mathbb{G}(\log v)dv \leq \beta_1 < 1. \tag{7}$$

$(\Sigma 3)$ There exists constants $\eta > 0$, $0 < \beta_2 < 1 - \beta_1$ and a CF $\mathcal{F} : [1, \infty) \times (0, \eta] \times (0, \eta] \to \mathbb{R}^+$ such that

$$\frac{|\mathcal{F}(\varsigma, v_1\mathbb{G}(\log\varsigma), v_2\mathbb{G}(\log(1 + \vartheta\varsigma)))|}{\mathbb{G}(\log\varsigma)} \leq \mathcal{F}(\varsigma, |v_1|, |v_2|), \tag{8}$$

holds for all $\varsigma \geq 1$, $0 < |v_1|, |v_2| \leq \eta$ and

$$\sup_{\varsigma \geq 1}\int_1^{\varsigma} \frac{\mathcal{K}(\log\frac{\varsigma}{v})}{\mathbb{G}(\log\frac{\varsigma}{v})}\frac{\mathcal{F}(v, \mathfrak{r}_1, \mathfrak{r}_2)}{\eta}\frac{dv}{v} \leq \beta_2 < 1 - \beta_1, \tag{9}$$

holds for every $0 < \mathfrak{r}_1, \mathfrak{r}_2 \leq \eta$, where $\mathcal{F}(\varsigma, \mathfrak{r}_1, \mathfrak{r}_2)$ is nondecreasing in $\mathfrak{r}_1$ and $\mathfrak{r}_2$ for fixed ς, $\mathcal{F}(\varsigma, \mathfrak{r}_1, \mathfrak{r}_2) \in L^1([1, +\infty))$ in ς for fixed $\mathfrak{r}_1$ and $\mathfrak{r}_2$, and

$$\mathcal{K}(\log\frac{\varsigma}{v}) = \begin{cases} \frac{1}{\Gamma(\alpha-1)}\int_v^{\varsigma} e^{-k(\varsigma-\tau)}\left(\log\frac{\tau}{v}\right)^{\alpha-2}\frac{d\tau}{\tau}, & \frac{\varsigma}{v} \geq 1, \\ 0, & \frac{\varsigma}{v} < 1. \end{cases} \tag{10}$$

Theorem 3. *Assume that* $(\Sigma 1) - (\Sigma 3)$ *hold. Then,* $\varphi = 0$ *of Equation (1) is stable in* $\mathcal{E}$.

Proof. First, we show the existence of $\delta > 0$, for all given $\varepsilon > 0$, such that

$$|\varphi_0| + |\varphi_1| < \delta \text{ implies } \|\varphi\| \leq \varepsilon.$$

Due to Equation (7), there is a constant $M_1 > 0$ as such

$$\frac{e^{-k\varsigma}}{\mathbb{G}(\log\varsigma)} \leq M_1. \tag{11}$$

Let $0 < \delta \leq \frac{\left(1-\beta_1\left(1+\frac{L_{\mathcal{H}}}{|k|}\right)-\beta_2\right)|k|}{M_1|k|+(1+M_1)(1+L_{\mathcal{H}})}\varepsilon$. Proposing the non-empty closed convex subset $\mathfrak{B}(\varepsilon) \subseteq \mathcal{E}$, for $\varsigma \geq 1$, we define the two mappings P and Q on $\mathfrak{B}(\varepsilon)$ as under

$$\begin{aligned} P\varphi(\varsigma) &= \frac{1}{\Gamma(\alpha-1)}\int_1^{\varsigma}\int_v^{\varsigma} e^{-k(\varsigma-\tau)}\left(\log\frac{\tau}{v}\right)^{\alpha-2} d\tau \mathcal{F}(v,\varphi(v),\varphi(1+\vartheta v))\frac{dv}{v} \\ &= \int_1^{\varsigma}\mathcal{K}(\log\frac{\varsigma}{v})\mathcal{F}(v,\varphi(v),\varphi(1+\vartheta v))\frac{dv}{v}, \end{aligned} \tag{12}$$

and

$$\begin{aligned} Q\varphi(\varsigma) &= \varphi_0 e^{-k\varsigma} + (\varphi_1 - \mathcal{H}(1,\varphi(1+\vartheta)))\int_1^{\varsigma} e^{-k(\varsigma-\tau)}\frac{d\tau}{\tau} \\ &+ \int_1^{\varsigma} e^{-k(\varsigma-\tau)}(k\tau\varphi(\tau) + \mathcal{H}(\tau,\varphi(1+\vartheta\tau)))\frac{d\tau}{\tau}. \end{aligned} \tag{13}$$

Clearly, for $\varphi \in \mathfrak{B}(\varepsilon)$, $P\varphi$ and $Q\varphi$ are CFs on $[1,+\infty)$. In addition, for $\varphi \in \mathfrak{B}(\varepsilon)$, by Equations (7)–(9) for any $\varsigma \geq 1$, we have

$$\begin{aligned} \frac{|P\varphi(\varsigma)|}{\mathbb{G}(\log\varsigma)} &\leq \int_1^{\varsigma}\frac{\mathcal{K}(\log\frac{\varsigma}{v})}{\mathbb{G}(\log\frac{\varsigma}{v})}\frac{|\mathcal{F}(v,\varphi(v),\varphi(1+\vartheta v))|}{\mathbb{G}(\log v)}\frac{dv}{v} \\ &\leq \int_1^{\varsigma}\frac{\mathcal{K}(\log\frac{\varsigma}{v})}{\mathbb{G}(\log\frac{\varsigma}{v})}\mathcal{F}\left(v,\frac{|\varphi(v)|}{\mathbb{G}(\log v)},\frac{|\varphi(1+\vartheta v)|}{\mathbb{G}(\log(1+\vartheta v))}\right)\frac{dv}{v} \\ &\leq \beta_2\|\varphi\| \leq \beta_2\varepsilon < \infty, \end{aligned} \tag{14}$$

and

$$\begin{aligned} \frac{|Q\varphi(\varsigma)|}{\mathbb{G}(\log\varsigma)} &= \Bigg|\varphi_0\frac{e^{-k\varsigma}}{\mathbb{G}(\log\varsigma)} + (\varphi_1 - \mathcal{H}(1,\varphi(1+\vartheta)))\frac{\int_1^{\varsigma} e^{-k(\varsigma-\tau)}\frac{d\tau}{\tau}}{\mathbb{G}(\log\varsigma)} \\ &+ \int_1^{\varsigma}\frac{e^{-k(\varsigma-\tau)}}{\mathbb{G}(\log\varsigma)}(k\tau\varphi(\tau) + \mathcal{H}(\tau,\varphi(1+\vartheta\tau)))\frac{d\tau}{\tau}\Bigg| \\ &\leq M_1|\varphi_0| + \frac{1+M_1}{|k|}(|\varphi_1| + |\mathcal{H}(1,\varphi(1+\vartheta))|) \\ &+ |k|\int_1^{\infty}\frac{e^{-kv}}{\mathbb{G}(\log v)}dv(1+\frac{L_{\mathcal{H}}}{|k|})\|\varphi\| \\ &\leq M_1|\varphi_0| + \frac{1+M_1}{|k|}(|\varphi_1| + L_{\mathcal{H}}|\varphi(\varsigma)|) + \beta_1\left(1+\frac{L_{\mathcal{H}}}{|k|}\right)\varepsilon < \infty. \end{aligned} \tag{15}$$

Therefore, we have $P\mathfrak{B}(\varepsilon) \subseteq \mathcal{E}$ and $Q\mathfrak{B}(\varepsilon) \subseteq \mathcal{E}$. Then, we utilize the Theorem 1 to demonstrate that there is at least one fixed point of the operator $P+Q$ in $\mathfrak{B}(\varepsilon)$. We now present the proof in three steps.

Step 1. Proving $P\varphi + Q\mathfrak{y} \in \mathfrak{B}(\varepsilon)$ for every $\varphi, \mathfrak{y} \in \mathfrak{B}(\varepsilon)$. By Equations (14) and (15), we have

$$\begin{aligned} \sup_{\varsigma\geq 1}\frac{|P\varphi(\varsigma) + Q\mathfrak{y}(\varsigma)|}{\mathbb{G}(\log\varsigma)} &= \sup_{\varsigma\geq 1}\Bigg\{\Bigg|\varphi_0\frac{e^{-k\varsigma}}{\mathbb{G}(\log\varsigma)} + (\varphi_1 - \mathcal{H}(1,\mathfrak{y}(1+\vartheta)))\frac{\int_1^{\varsigma} e^{-k(\varsigma-\tau)}\frac{d\tau}{\tau}}{\mathbb{G}(\log\varsigma)} \\ &+ \int_1^{\varsigma}\frac{e^{-k(\varsigma-\tau)}}{\mathbb{G}(\log\varsigma)}(k\tau\mathfrak{y}(\tau) + \mathcal{H}(\tau,\mathfrak{y}(1+\vartheta\tau)))\frac{d\tau}{\tau} \\ &+ \int_1^{\varsigma}\frac{\mathcal{K}(\log\frac{\varsigma}{v})}{\mathbb{G}(\log\varsigma)}\mathcal{F}(v,\varphi(v),\varphi(1+\vartheta v))\frac{dv}{v}\Bigg|\Bigg\} \\ &\leq M_1|\varphi_0| + \frac{1+M_1}{|k|}(|\varphi_1| + L_{\mathcal{H}}\delta) \\ &+ |k|\int_1^{\infty}\frac{e^{-kv}}{\mathbb{G}(\log v)}dv\left(1+\frac{L_{\mathcal{H}}}{|k|}\right)\|\mathfrak{y}\| + \beta_2\|\varphi\| \\ &\leq \frac{M_1|k| + (1+M_1)(1+L_{\mathcal{H}})}{|k|}\delta + \beta_1\left(1+\frac{L_{\mathcal{H}}}{|k|}\right)\varepsilon + \beta_2\varepsilon \leq \varepsilon, \end{aligned}$$

which conclude that $P\varphi + Q\mathfrak{y} \in \mathfrak{B}(\varepsilon)$ for all $\varphi, \mathfrak{y} \in \mathfrak{B}(\varepsilon)$.

Step 2. Now we just show that $P\mathfrak{B}(\varepsilon)$ is relatively compact in $\mathcal{E}$, to see that P is continuous. In fact, from Equation (14), we obtain that $\{\varphi(\varsigma)/\mathbb{G}(\log\varsigma) : \varphi \in \mathfrak{B}(\varepsilon)\}$ is uniformly bounded in $\mathcal{E}$. In addition, the usual theory says that the convolution of an L^1-function with a function tending to zero. So, we summarize that for $\frac{\varsigma}{v} \geq 1$, we obtain

$$\begin{aligned} 0 &\leq \lim_{\varsigma\to\infty} \frac{\mathcal{K}(\log\frac{\varsigma}{v})}{\mathbb{G}(\log\frac{\varsigma}{v})} \leq \lim_{\varsigma\to\infty} \frac{1}{\Gamma(\alpha-1)} \int_v^{\varsigma} \frac{e^{-k(\varsigma-\tau)}}{\mathbb{G}(\log\frac{\varsigma}{v})} \frac{(\log\frac{\tau}{v})^{\alpha-2}}{\mathbb{G}(\log\frac{\tau}{v})} \frac{d\tau}{\tau} \\ &= \lim_{\varsigma\to\infty} \frac{1}{\Gamma(\alpha-1)} \int_1^{\varsigma} \frac{e^{-k(\varsigma-v\tau)}}{\mathbb{G}(\log\frac{\varsigma}{v\tau})} \frac{(\log\tau)^{\alpha-2}}{\mathbb{G}(\log\tau)} \frac{d\tau}{\tau} = 0, \end{aligned} \tag{16}$$

due to the fact $\lim_{\varsigma\to\infty} \frac{(\log\varsigma)^{\alpha-2}}{\mathbb{G}(\log\varsigma)} = 0$. With continuity of $\mathcal{K}$ and $\mathbb{G}$, we find that there is a constant $M_2 > 0$ as such

$$\left| \frac{\mathcal{K}(\log\frac{\varsigma}{v})}{\mathbb{G}(\log\frac{\varsigma}{v})} \right| \leq M_2, \tag{17}$$

and for any $T_0 \in [1,\infty)$, the function $\mathcal{K}(\log\frac{\varsigma}{v})\mathbb{G}(\log v)/\mathbb{G}(\log\varsigma)$ is uniformly continuous on $\{(\varsigma, v) : 1 \leq v \leq \varsigma \leq T_0\}$. For any $\varsigma_1, \varsigma_2 \in [1, T_0]$, $\varsigma_1 < \varsigma_2$, we have

$$\begin{aligned} \left| \frac{P\varphi(\varsigma_2)}{\mathbb{G}(\log\varsigma_2)} - \frac{P\varphi(\varsigma_1)}{\mathbb{G}(\log\varsigma_1)} \right| &= \left| \int_1^{\varsigma_2} \frac{\mathcal{K}(\log\frac{\varsigma_2}{v})}{\mathbb{G}(\log\varsigma_2)} \mathcal{F}(v,\varphi(v),\varphi(1+\vartheta v)) \frac{dv}{v} \right. \\ &\quad \left. - \int_1^{\varsigma_1} \frac{\mathcal{K}(\log\frac{\varsigma_1}{v})}{\mathbb{G}(\log\varsigma_1)} \mathcal{F}(v,\varphi(v),\varphi(1+\vartheta v)) \frac{dv}{v} \right| \\ &\leq \int_1^{\varsigma_1} \left| \frac{\mathcal{K}(\log\frac{\varsigma_2}{v})}{\mathbb{G}(\log\varsigma_2)} - \frac{\mathcal{K}(\log\frac{\varsigma_1}{v})}{\mathbb{G}(\log\varsigma_1)} \right| |\mathcal{F}(v,\varphi(v),\varphi(1+\vartheta v))| \frac{dv}{v} \\ &\quad + \int_{\varsigma_1}^{\varsigma_2} \frac{\mathcal{K}(\log\frac{\varsigma_2}{v})}{\mathbb{G}(\log\frac{\varsigma_2}{v})} \mathcal{F}(v,\varepsilon,\varepsilon) \frac{dv}{v} \\ &\leq \int_1^{\varsigma_1} \left| \frac{\mathcal{K}(\log\frac{\varsigma_2}{v})\mathbb{G}(\log v)}{\mathbb{G}(\log\varsigma_2)} - \frac{\mathcal{K}(\log\frac{\varsigma_1}{v})\mathbb{G}(\log v)}{\mathbb{G}(\log\varsigma_1)} \right| \mathcal{F}(v,\varepsilon,\varepsilon) \frac{dv}{v} \\ &\quad + M_2 \int_{\varsigma_1}^{\varsigma_2} \mathcal{F}(v,\varepsilon,\varepsilon) \frac{dv}{v} \ \to \ 0, \end{aligned}$$

as $\varsigma_2 \to \varsigma_1$, which means that $\{\varphi(\varsigma)/\mathbb{G}(\log\varsigma) : \varphi \in \mathfrak{B}(\varepsilon)\}$ is equicontinuous. By Theorem 2, to prove that $P\mathfrak{B}(\varepsilon)$ is a relatively compact set of $\mathcal{E}$. It is enough just to prove $\{\varphi(\varsigma)/\mathbb{G}(\log\varsigma)\colon \varphi \in \mathfrak{B}(\varepsilon)\}$ is equiconvergent at infinity. For this, for every $\varepsilon_1 > 0$, there is $L > 1$ as such

$$M_2 \int_L^{\infty} \mathcal{F}(v,\varepsilon,\varepsilon) \frac{dv}{v} \leq \frac{\varepsilon_1}{3}.$$

From Equation (16), we obtain

$$\lim_{\varsigma\to\infty} \sup_{v\in[1,L]} \frac{\mathcal{K}(\log\frac{\varsigma}{v})}{\mathbb{G}(\log\frac{\varsigma}{v})} \leq \max\left\{ \lim_{\varsigma\to\infty} \frac{\mathcal{K}(\log\frac{\varsigma}{L})}{\mathbb{G}(\log\frac{\varsigma}{L})}, \lim_{\varsigma\to\infty} \frac{\mathcal{K}(\log\varsigma)}{\mathbb{G}(\log\varsigma)} \right\} = 0.$$

Thus, there exists $T > L$ as such $\varsigma_1, \varsigma_2 \geq T$, we obtain

$$\begin{aligned} \sup_{v\in[1,L]} \left| \frac{\mathcal{K}(\log\frac{\varsigma_2}{v})\mathbb{G}(\log v)}{\mathbb{G}(\log\varsigma_2)} - \frac{\mathcal{K}(\log\frac{\varsigma_1}{v})\mathbb{G}(\log v)}{\mathbb{G}(\log\varsigma_1)} \right| &\leq \sup_{v\in[1,L]} \left| \frac{\mathcal{K}(\log\frac{\varsigma_2}{v})}{\mathbb{G}(\log\frac{\varsigma_2}{v})} \right| + \sup_{v\in[1,L]} \left| \frac{\mathcal{K}(\log\frac{\varsigma_1}{v})}{\mathbb{G}(\log\frac{\varsigma_1}{v})} \right| \\ &\leq \frac{\varepsilon_1}{3} \left(\int_1^{\infty} \mathcal{F}(v,\varepsilon,\varepsilon) \frac{dv}{v} \right)^{-1}. \end{aligned}$$

Consequently, for $\varsigma_1, \varsigma_2 \geq T$,

$$\begin{aligned}\left|\frac{P\varphi(\varsigma_2)}{\mathbb{G}(\log\varsigma_2)}-\frac{P\varphi(\varsigma_1)}{\mathbb{G}(\log\varsigma_1)}\right| &= \left|\int_1^{\varsigma_2}\frac{\mathcal{K}(\log\frac{\varsigma_2}{v})}{\mathbb{G}(\log\varsigma_2)}\mathcal{F}(v,\varphi(v),\varphi(1+\vartheta v))\frac{dv}{v}\right.\\ &\quad\left.-\int_1^{\varsigma_1}\frac{\mathcal{K}(\log\frac{\varsigma_1}{v})}{\mathbb{G}(\log\varsigma_1)}\mathcal{F}(v,\varphi(v),\varphi(1+\vartheta v))\frac{dv}{v}\right|\\ &\leq \int_1^{L}\left|\frac{\mathcal{K}(\log\frac{\varsigma_2}{v})\mathbb{G}(\log v)}{\mathbb{G}(\log\varsigma_2)}-\frac{\mathcal{K}(\log\frac{\varsigma_1}{v})\mathbb{G}(\log v)}{\mathbb{G}(\log\varsigma_1)}\right|\mathcal{F}(v,\varepsilon,\varepsilon)\frac{dv}{v}\\ &\quad+\int_L^{\varsigma_2}\frac{\mathcal{K}(\log\frac{\varsigma_2}{v})}{\mathbb{G}(\log\frac{\varsigma_2}{v})}\mathcal{F}(v,\varepsilon,\varepsilon)\frac{dv}{v}+\int_L^{\varsigma_1}\frac{\mathcal{K}(\log\frac{\varsigma_1}{v})}{\mathbb{G}(\log\frac{\varsigma_1}{v})}\mathcal{F}(v,\varepsilon,\varepsilon)\frac{dv}{v}\\ &\leq \frac{\varepsilon_1}{3}+2M_2\int_L^{\infty}\mathcal{F}(v,\varepsilon,\varepsilon)\frac{dv}{v}\leq\varepsilon_1.\end{aligned}$$

Therefore, the necessary conclusion is achieved.

Step 3. Let us say that $Q : \mathfrak{B}(\varepsilon) \to \mathcal{E}$ is a contraction mapping. In fact, for any $\varphi, \mathfrak{y} \in \mathfrak{B}(\varepsilon)$, from Equation (7), we find that

$$\begin{aligned}\sup_{\varsigma\geq1}\left|\frac{Q\varphi(\varsigma)}{\mathbb{G}(\log\varsigma)}-\frac{Q\mathfrak{y}(\varsigma)}{\mathbb{G}(\log\varsigma)}\right| &= \sup_{\varsigma\geq1}\left\{\left|\int_1^{\varsigma}\frac{e^{-k(\varsigma-v)}}{\mathbb{G}(\log\varsigma)}(kv\varphi(v)+\mathcal{H}(v,\varphi(1+\vartheta v)))\frac{dv}{v}\right.\right.\\ &\quad\left.\left.-\int_1^{\varsigma}\frac{e^{-k(\varsigma-v)}}{\mathbb{G}(\log\varsigma)}(kv\mathfrak{y}(v)+\mathcal{H}(v,\mathfrak{y}(1+\vartheta v)))\frac{dv}{v}\right|\right\}\\ &\leq \sup_{\varsigma\geq1}|k|\int_1^{\varsigma}\frac{e^{-k(\varsigma-v)}}{\mathbb{G}(\log\frac{\varsigma}{v})}\frac{v|\varphi(v)-\mathfrak{y}(v)|}{\mathbb{G}(\log v)}\frac{dv}{v}\\ &\quad+\sup_{\varsigma\geq1}\int_1^{\varsigma}\frac{e^{-k(\varsigma-v)}}{\mathbb{G}(\log\frac{\varsigma}{v})}\frac{|\mathcal{H}(v,\varphi(1+\vartheta v))-\mathcal{H}(v,\mathfrak{y}(1+\vartheta v))|}{\mathbb{G}(\log v)}\frac{dv}{v}\\ &\leq |k|\int_1^{\varsigma}\frac{e^{-k(\varsigma-v)}}{\mathbb{G}(\log\frac{\varsigma}{v})}dv\left(1+\frac{L_{\mathcal{H}}}{|k|}\right)\|\varphi-\mathfrak{y}\|\\ &\leq \beta_1\left(1+\frac{L_{\mathcal{H}}}{|k|}\right)\|\varphi-\mathfrak{y}\|.\end{aligned}$$

By Theorem 1, we are aware that the operator $P+Q$ has at least one fixed point in $\mathfrak{B}(\varepsilon)$. Finally, for any $\varepsilon_2 > 0$, if $0 < \delta_1 \leq \frac{\left(1-\beta_1\left(1+\frac{L_{\mathcal{H}}}{|k|}\right)-\beta_2\right)|k|}{|k|M_1+(1+M_1)(1+L_{\mathcal{H}})}\varepsilon_2$, then $|\varphi_0|+|\varphi_1| \leq \delta_1$ implies that

$$\begin{aligned}\|\varphi\| = \sup_{\varsigma\geq1}&\left|\varphi_0\frac{e^{-k\varsigma}}{\mathbb{G}(\log\varsigma)}+(\varphi_1-\mathcal{H}(1,\varphi(1+\vartheta)))\frac{\int_1^{\varsigma}e^{-k(\varsigma-\tau)}\frac{d\tau}{\tau}}{\mathbb{G}(\log\varsigma)}\right.\\ &+\int_1^{\varsigma}\frac{e^{-k(\varsigma-\tau)}}{\mathbb{G}(\log\varsigma)}(kv\varphi(v)+\mathcal{H}(v,\varphi(1+\vartheta v)))\frac{dv}{v}\\ &\left.+\int_1^{\varsigma}\frac{\mathcal{K}(\log\frac{\varsigma}{v})}{\mathbb{G}(\log\varsigma)}\mathcal{F}(v,\varphi(v),\varphi(1+\vartheta v))\frac{dv}{v}\right|\end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{\varsigma\geq 1}\Bigg\{\frac{e^{-k\varsigma}}{\mathbb{G}(\log\varsigma)}\varphi_0+\frac{\left|1-e^{-k\varsigma}\right|}{|k|\mathbb{G}(\log\varsigma)}(|\varphi_1|+L_{\mathcal{H}}|\varphi(1+\vartheta)|)\\
&+|k|\int_1^{\varsigma}\frac{e^{-k(\varsigma-v)}}{\mathbb{G}(\log\frac{\varsigma}{v})\mathbb{G}(\log v)}\left(v|\varphi(v)|+\frac{L_{\mathcal{H}}}{|k|}|\varphi(v)|\right)\frac{dv}{v}\\
&+\int_1^{\varsigma}\frac{\mathcal{K}(\log\frac{\varsigma}{v})}{\mathbb{G}(\log\frac{\varsigma}{v})}\frac{|\mathcal{F}(v,\varphi(v),\varphi(1+\vartheta v))|}{\mathbb{G}(\log v)}\frac{dv}{v}\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&\leq M_1\delta_1+\frac{1+M_1}{|k|}(\delta_1+L_{\mathcal{H}}\delta_1)+\beta_1\left(1+\frac{L_{\mathcal{H}}}{|k|}\right)\|\varphi\|+\beta_2\|\varphi\|\\
&\leq\frac{|k|M_1+(1+M_1)(1+L_{\mathcal{H}})}{\left(1-\beta_1\left(1+\frac{L_{\mathcal{H}}}{|k|}\right)-\beta_2\right)|k|}\delta_1\leq\varepsilon_2.
\end{aligned}
$$

Therefore, we assert that the trivial solution of Equation (1) is stable in $\mathcal{E}$. □

Theorem 4. *Let all assumptions of Theorem 3 be verified,*

$$\lim_{\varsigma\to\infty}e^{-k\varsigma}/\mathbb{G}(\log\varsigma)=0, \tag{18}$$

and for any $\mathfrak{r}>0$, there exists a function $\psi_{\mathfrak{r}}(\varsigma)\in L^1([1,+\infty))$, $\psi_{\mathfrak{r}}(\varsigma)>0$ as such $|v|,|v|\leq\mathfrak{r}$ implies

$$|\mathcal{F}(\varsigma,v,v)|/\mathbb{G}(\log\varsigma)\leq\psi_{\mathfrak{r}}(\varsigma),\ \textit{a.e. }\varsigma\in[1,+\infty). \tag{19}$$

Then the trivial solution of Equation (1) is AS.

Proof. By virtue of Theorem 3, the trivial solution of Equation (1) is stable. Next, we demonstrate that the trivial solution $\varphi=0$ of Equation (1) is attractive.

For any $\mathfrak{r}>0$, defining

$$\mathfrak{B}_*(\mathfrak{r})=\left\{\varphi\in\mathfrak{B}(\mathfrak{r}),\ \lim_{\varsigma\to\infty}\varphi(\varsigma)/\mathbb{G}(\log\varsigma)=0\right\}.$$

We need to prove that $P\varphi+Q\mathfrak{y}\in\mathfrak{B}_*(\mathfrak{r})$ for every $\varphi,\mathfrak{y}\in\mathfrak{B}_*(\mathfrak{r})$, i.e.,

$$\frac{P\varphi(\varsigma)+Q\mathfrak{y}(\varsigma)}{\mathbb{G}(\log\varsigma)}\to 0\text{ as }\varsigma\to\infty,$$

where

$$
\begin{aligned}
P\varphi(\varsigma)+Q\mathfrak{y}(\varsigma)&=\varphi_0e^{-k\varsigma}+(\varphi_1-\mathcal{H}(1,\mathfrak{y}(1+\vartheta)))\frac{\int_1^{\varsigma}e^{-k(\varsigma-\tau)}\frac{d\tau}{\tau}}{\mathbb{G}(\log\varsigma)}\\
&+\int_1^{\varsigma}e^{-k(\varsigma-\tau)}(kv\mathfrak{y}(v)+\mathcal{H}(v,\mathfrak{y}(1+\vartheta v)))\frac{dv}{v}\\
&+\int_1^{\varsigma}\mathcal{K}(\log\frac{\varsigma}{v})\mathcal{F}(v,\varphi(v),\varphi(1+\vartheta v))\frac{dv}{v}.
\end{aligned}
$$

From Equations (7) and (18) and for φ, $\mathfrak{y}\in\mathfrak{B}_*(\mathfrak{r})$, we have

$$\int_1^{\varsigma}\frac{e^{-k(\varsigma-v)}}{\mathbb{G}(\log\frac{\varsigma}{v})}\frac{(kv\mathfrak{y}(v)+\mathcal{H}(v,\mathfrak{y}(1+\vartheta v))}{\mathbb{G}(\log v)}\frac{dv}{v}\to 0,\ \text{as }\varsigma\to\infty$$

and

$$\frac{\mathcal{K}(\log\frac{\varsigma}{v})}{\mathbb{G}(\log\frac{\varsigma}{v})}=\frac{\int_v^{\varsigma}\frac{e^{-k(\varsigma-\tau)}}{\mathbb{G}(\log\frac{\varsigma}{v})}\left(\log\frac{\tau}{v}\right)^{\alpha-2}\frac{d\tau}{\tau}}{\Gamma(\alpha-1)}\to 0,\ \text{as }\varsigma\to\infty.$$

By the hypothesis that $\psi_{\mathfrak{r}}(\varsigma) \in L^1([1, +\infty))$, we find that

$$\int_1^{\varsigma} \frac{\mathcal{K}(\log\frac{\varsigma}{v})}{\mathbb{G}(\log\frac{\varsigma}{v})} \frac{|\mathcal{F}(v, \varphi(v), \varphi(1+\vartheta v))|}{\mathbb{G}(\log v)} \frac{dv}{v} \leq \int_1^{\varsigma} \frac{\mathcal{K}(\log\frac{\varsigma}{v})}{\mathbb{G}(\log\frac{\varsigma}{v})} \psi_{\mathfrak{r}}(v) \frac{dv}{v} \to 0,$$

as $\varsigma \to \infty$. We obtain the conclusion. $\square$

4. An Application

Consider the following equation in frame of CFD, which is a particular case of CHFD:

$$\begin{cases} {}^{C}\mathcal{D}_{0+}^{\frac{3}{2}}\varphi(\varsigma) = \delta\frac{\varsigma^2\varphi^2}{e^{(\lambda+1)\varsigma}} + \frac{\delta\varphi\sqrt{\varphi}}{(1+\varsigma^2)e^{\lambda\varsigma/2}}, \\ \varphi(0) = \varphi_0, \varphi'(0) = \varphi_1, \end{cases} \tag{20}$$

where $\lambda > 1, \delta > 0$ and $\alpha = \frac{3}{2}$. Let $0 < |k| \leq \frac{\lambda-1}{2}$, $\mathbb{G}(\varsigma) = e^{\lambda\varsigma}$, and $\beta_1 = \frac{|k|}{\lambda+k}$. Then, Equation (7) holds.

It follows that the Banach space is

$$\mathcal{E} = \left\{\varphi(\varsigma) \in C[0, +\infty) : \sup_{\varsigma\geq 0}|\varphi(\varsigma)|/e^{\lambda\varsigma} < \infty\right\},$$

equipped with the norm $\|\varphi\| = \sup_{\varsigma\geq 0}\frac{|\varphi(\varsigma)|}{e^{\lambda\varsigma}}$. Let $\tilde{\mathcal{F}}(\varsigma, \mathfrak{r}) = \delta\mathfrak{r}^2\varsigma^2 e^{-\varsigma} + \frac{\delta\mathfrak{r}\sqrt{\mathfrak{r}}}{1+\varsigma^2}$. Then, Equation (8) holds and $\tilde{\mathcal{F}}(\varsigma, \mathfrak{r}) \in L^1[0, +\infty)$ in ς for fixed $\mathfrak{r}$. Note that

$$\begin{aligned} \frac{\mathcal{K}(\varsigma-u)}{e^{\lambda(\varsigma-u)}} &= \frac{1}{\Gamma(1/2)}\int_u^{\varsigma} \frac{1}{\sqrt{s-u}e^{\lambda(s-u)}} \frac{1}{e^{(\lambda+k)(\varsigma-s)}} ds \\ &\leq \frac{\int_u^{\varsigma}\frac{1}{\sqrt{s-u}e^{\lambda(s-u)}}ds}{\Gamma(1/2)} = \frac{\int_0^{\varsigma-u}\frac{1}{\sqrt{\sigma}e^{\lambda\sigma}}ds}{\Gamma(1/2)} \leq \sqrt{\lambda}, \end{aligned}$$

for all $\varsigma \geq 0$, if there exists $\eta \geq 0$ such that

$$\delta \leq \frac{1}{2(2\eta + \frac{\pi}{2}\eta^{1/2})(\lambda+k)\lambda^{1/2} + 1}, \tag{21}$$

then

$$\int_0^{\varsigma} \frac{\mathcal{K}(\varsigma-u)}{\mathbb{G}(\varsigma-u)} \frac{\tilde{f}(u, \mathfrak{r})}{\mathfrak{r}} du = \delta\int_0^{\varsigma} \frac{\mathcal{K}(\varsigma-u)}{\mathbb{G}(\varsigma-u)}(\mathfrak{r}\varsigma^2 e^{-\varsigma} + \frac{\mathfrak{r}^{1/2}}{1+\varsigma^2})du \leq \frac{1/2}{\lambda+k} < 1-\beta_1,$$

for all $\varsigma \geq 0, 0 \leq \mathfrak{r} \leq \eta$. Thus, the trivial solution of Equation (20) is stable in $\mathcal{E}^*$, which follows from Theorem 3.

Moreover, if we let $\varphi_{\mathfrak{r}}(\varsigma) = \delta\frac{\varsigma^2\mathfrak{r}^2}{e^{(\lambda+1)\varsigma}} + \frac{\delta\mathfrak{r}\sqrt{\mathfrak{r}}}{(1+\varsigma^2)e^{\lambda\varsigma/2}} \in L^1[0, +\infty)$, then for any bounded $\mathfrak{r} > 0$, we find that $|\mathcal{F}(\varsigma, u)| \leq \varphi_{\mathfrak{r}}(\varsigma)$ and

$$\lim_{\varsigma\to\infty} e^{-k\varsigma}/\mathbb{G}(\varsigma) \leq \lim_{\varsigma\to\infty} e^{-\frac{\lambda\varsigma}{2}} = 0,$$

which implies that the trivial solution of Equation (20) is AS, by the conclusion of Theorem 4.

5. Conclusions

In this paper, a second-order pantograph fractional differential equation in weighted Banach space has been studied. The considered equation is of the second-order, framed in terms of the Caputo–Hadamard fractional operator, and designed in a general form so it covers several particular cases. The AS of the trivial solution of the main equation has been provided. By using the KFPT and turning Equation (1) into an analogous integral equation, the main theorem was proved in a different way than the usual procedures.

The findings can be viewed as adding to the body of knowledge because they describe less-frequent sorts of equations. The same strategy can be used to demonstrate higher order pantograph fractional differential equations of hybrid type or higher-order pantograph fractional differential equations with impulsive effects. We leave these ideas for future consideration.

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