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Hermite wavelets collocation method for solving a fredholm integro-differential equation with fractional Caputo-Fabrizio derivative

Youcef Henka
Université 8 Mai 1945 Guelma, Algérie
Samir Lemita
Ecole Normale Supérieure-Ouargla, Algérie
and
Mohamed Zine Aissaoui
Université 8 Mai 1945 Guelma, Algérie
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Abstract

In this paper, we investigate the numerical study of nonlinear Fredholm integro-differential equation with the fractional Caputo-Fabrizio derivative. We use the Hermite wavelets and collocation technique to approximate the exact solution by reducing the Fredholm integro-differential equation to a nonlinear algebraic system. Furthermore, we apply this numerical method on certain examples to check its accuracy and validity.

Keywords: *Fredholm integro-differential equation, Nonlinear equation, Collocation technique, Hermite Wavelets, Fractional derivative.*

MSC (2010): *26A33, 45J05, 65T60.*

1. Introduction

Recently, the integro-differential equations have represented a great interest among the different types of mathematical equations. This is due to its use in the modeling of various problems in several fields of science: electrostatics, fluid dynamics, scattering, engineering, biology and medicine such as [7, 12, 16, 13]. Hence, there are many publications that are interested on the analytical and numerical studies for the integro differential equations. We find in [2, 9], the authors studied the solvability of linear Fredholm integral equation with weakly singular kernel. The authors, in [3, 15], examined the nonlinear Fredholm integro-differential equation when the derivative of the unknown function is inside of the integral operator. On the other hands, many numerical methods are applied to get an approximate solution for such equations, Laplace decomposition method [1], Rational approximation [11], B-spline method [18, 14], Homotopy perturbation method [20], Modified variational iteration method [10], CAS wavelet operational matrix [6, 19].

Also, in [8], the author studied the existence and uniqueness of the following nonlinear Volterra integro-differential equation with Caputo derivative

$$u(x) = f(x) + \int_a^x K\left(x, s, u(s), {}^{C}\mathcal{D}^{\alpha}u(s)\right) ds, \quad \forall x \in [a, b], \tag{1.1}$$

where u is the unknown function and ${}^{C}\mathcal{D}^{\alpha}$ is the standard Caputo fractional derivative of order $0 < \alpha < 1$. the product integration method was applied to approximate its exact solution.

In [4], the authors used the wavelet collocation method to solve the following fractional Fredholm integro-differential equation

$$\begin{aligned} &{}^{C}\mathcal{D}^{\alpha}u(x) = c_1 f(x, u(x)) + c_2 \int_0^1 K(x,t)g(u(t))dt, \quad x \in [0,1], \\ &u(0) = u_0, \quad c_1, c_2 \in \mathbf{R}. \end{aligned} \tag{1.2}$$

where ${}^{C}\mathcal{D}^{\alpha}$ denote to the Caputo fractional derivative of order α. f is assumed to be a sufficiently smooth function on $[0,1] \times \mathbf{R}$, and K is a continous function on $[0,1]^2$.

In this article, we focus on a numerical study for the following fractional nonlinear Fredholm integro-differential equation:

$$\begin{aligned} u(x) &= g(x) + \int_0^1 K(x, s, u(s), \mathcal{D}^\alpha u(s))ds \\ u(0) &= 0, \end{aligned} \tag{1.3}$$

where, $0 < \alpha < 1$, $u(x), g(x) \in H^1[0,1]$, $K, \partial_x K \in C([0,1]^2 \times \mathbf{R}^2)$ and $\mathcal{D}^\alpha$ denote the fractional Caputo Fabrizio derivative of order α described in [5].

This paper is organized as follow: In the second section we introduce some necessary definitions and theorems which we need. In the third section we give the properties of Hermite wavelets. In the fourth section, we construct the operational matrix of fractional integration. In fifth section, we describe our suggested method, and in the last section, we check the validity and efficiency of our method by some illustrative examples.

2. Preliminaries

We introduce some necessary definitions which will be used in the sequel.

Definition 1. *Let $\alpha \in \mathbf{R}$, such that $0 < \alpha < 1$. The Caputo-Fabrizio fractional derivative of order α of a function $u \in H^1[0,1]$ is defined by [5]:*

$$\mathcal{D}^\alpha u(x) = \frac{1}{1-\alpha} \int_0^x e^{-\frac{\alpha}{1-\alpha}(x-s)} u'(s) ds.$$

Definition 2. *Let $\alpha \in \mathbf{R}$ such that $0 < \alpha < 1$. The Caputo-Fabrizio fractional integral of order α of a function $u \in H^1[0,1]$ is defined by [17]:*

$$\mathcal{I}^\alpha u(x) = (1-\alpha) u(x) + \alpha \int_0^x u(s) ds.$$

Lemma 1. *Let be $\alpha \in \mathbf{R}$, such that $0 < \alpha < 1$, and $u \in H^1([0,1])$, then we have the following equalities:*

$$\begin{aligned} \mathcal{D}^\alpha \left(\mathcal{I}^\alpha u(x) \right) &= u(x) - e^{-\frac{\alpha}{1-\alpha}(x-a)} \cdot u(0), \\ \mathcal{I}^\alpha \left(\mathcal{D}^\alpha u(x) \right) &= u(x) - u(0). \end{aligned}$$

Proof. See [17]. □

From the previous lemma and, there is minor drawback in Caputo-Fabrizio fractional derivative, the equality $\mathcal{D}^{\alpha}\left(\mathcal{I}^{\alpha}u(x)\right) = \mathcal{I}^{\alpha}\left(\mathcal{D}^{\alpha}u(x)\right)$ is not hold true in general cases.

3. Properties of Hermite Wavelets

3.1. Hermite Polynomial

The Hermite polynomials are orthogonal with respect to the weight function $w(x) = \sqrt{1-x^2}$ on $\mathbf{R}$, and satisfy the following recurrence formula:

$$\begin{cases} H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \\ H_1(x) = 2x, \quad H_0(x) = 1, \qquad \text{for} \quad n \geq 1. \end{cases}$$

The first few Hermite polynomials are:

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = 4x^2 - 2, \quad H_3(x) = 8x^3 - 12x$$

$$H_4(x) = 16x^4 - 48x^2 + 12, \quad H_5(x) = 32x^5 - 160x^3 + 120x$$

3.2. Hermite Wavelets

The wavelet functions are constructed from a dilation parameter α, and translation parameter β varies continuously from a single function called the mother wavelet, then we have the following family of continuous wavelets :

$$\psi_{\alpha,\beta}(x) = \frac{1}{\sqrt{|\alpha|}} H\left(\frac{x-\beta}{\alpha}\right), \qquad \alpha, \beta \in \mathbf{R}, \quad \alpha \neq 0.$$

Hermite wavelets are defined as :

$$\psi_{i,j}(x) = \begin{cases} \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} H_j(2^k x - 2i + 1), & \frac{i-1}{2^{k-1}} \leq x \leq \frac{i}{2^{k-1}}, \\ 0, & \text{otherwise.} \end{cases}$$

where $j = 0, 1, 2, \ldots, n-1$, and $i = 1, 2, \ldots, 2^{k-1}$. and k is a positive integer number, and H_j is a Hermite polynomial of degree j, then family of Hermite wavelets $\{\psi_{i,j}\}$ defines an orthonormal basis for $L^2_w(\mathbf{R})$.
For any function $u(x)$ in $L^2_w(\mathbf{R})$ can be written :

$$u(x) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} c_{i,j}\psi_{i,j}(x) \tag{3.1}$$

where

$c_{i,j} = \langle u, \psi_{i,j} \rangle$, such that $\langle ., . \rangle$ is the inner product in $L^2_w(\mathbf{R})$.
So we approximate the function $u(x)$ by truncated the infinite series (3.1) as follow:

$$u_n(x) = \sum_{i=1}^{2^{k-1}} \sum_{j=0}^{n-1} c_{i,j}\psi_{i,j}(x) = C^T P(x) \tag{3.2}$$

where C^T and $P(x)$ are $2^{k-1}n \times 1$ matrices:

$$C^T = [c_{1,0}, c_{1,1}, ..., c_{1,n-1}, c_{1,0}, c_{2,1}, ..., c_{2,n-1}, ..., c_{2^{k-1},0}, ..., c_{2^{k-1},n-1}],$$

and

$$P(x) = [\psi_{1,0}, \psi_{1,1}, ..., \psi_{1,n-1}, \psi_{2,0}, \psi_{2,1}, ..., \psi_{2,n-1}, ..., \psi_{2^{k-1},0}, \psi_{2^{k-1},1} ..., \psi_{2^{k-1},n-1}]^T.$$

4. Operational Matrix of Fractional Integration

Let $k = 1$, then both C^T and $P(x)$ would be:

$$\begin{aligned} C^T &= [\alpha_0, \alpha_1, ..., \alpha_{n-1}], \\ P(x) &= [\psi_0(x), \psi_1(x), ..., \psi_{n-1}(x)]. \end{aligned}$$

Let W_n be a matrix that contains the coefficients of Hermite wavelets:

$$W_n = \frac{1}{\sqrt{\pi}} \begin{pmatrix} 2 & -4 & 4 & \cdots \\ H_{n-1}(-1) & & & \\ 0 & 8 & -32 & \cdots \\ \vdots & & & \\ 0 & 0 & 32 & \cdots \\ \vdots & & & \\ \vdots & \vdots & \vdots & \ddots \\ \vdots & & & \\ 0 & 0 & 0 & \cdots \\ 2^{2n-1} & & & \end{pmatrix}$$

and

$$X_n(x) = (1, x, x^2, ..., x^{n-1}) \quad , \quad P_n(x) = [\psi_0(x), \psi_1(x), ..., \psi_{n-1}(x)]$$

So, we have:

$$P_n(x) = X_n(x)W_n.$$

Let N be an integral matrix in classical basis for polynomial space:

$$N = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{n} \end{pmatrix}$$

Then we have the following operational matrix of integration in the Hermite wavelets basis:

$$\int_0^x C^T P_n(y)dy = C^T M P_{n+1}(x),$$

where,

$$M = W_n^{-1} N W_{n+1}.$$

As well as, and by using the previous definition 2, we can represent the operational matrix of fractional integration as follow:

$$\mathcal{I}^\alpha \left(C^T P_n(x) \right) = C^T \left[(1-\alpha)F + \alpha M \right] P_{n+1}(x) = C^T Q_n(x), \tag{4.1}$$

such that $Q_n(x) = \left[(1-\alpha)F + \alpha M \right] P_{n+1}(x)$, and

$$F = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

5. Description of Method

Let the following Fredholm integro-differential equation:

$$\begin{cases} u(x) = f(x) + \displaystyle\int_0^1 K(x, s, u(s), \mathcal{D}^\alpha u(s))dy, \\ u(0) = 0, \end{cases} \tag{5.1}$$

we derive both of sides of equation (5.1) using the fractional Caputo-Fabrizio derivative of order α, then we get:

$$\mathcal{D}^{\alpha}u(x) = \mathcal{D}^{\alpha}g(x) + \int_0^1 \mathcal{D}_x^{\alpha}K(x, s, u(s), \mathcal{D}^{\alpha}u(s))ds. \tag{5.2}$$

We approach the unknown function $\mathcal{D}^{\alpha}u(x)$ by using the formula (3.2):

$$\mathcal{D}^{\alpha}u(x) \approx C^T P_n(x). \tag{5.3}$$

We integrate (5.3) by using the operational matrix of fractional integration described above (4.1), in order to obtain an approximation of unknown function $u(x)$ by:

$$u_n(x) \approx C^T Q_n(x). \tag{5.4}$$

Now, substitute (5.3) and (5.4) in (5.1), we obtain:

$$C^T P_n(x) = \mathcal{D}^{\alpha}g(x) + \int_0^1 \mathcal{D}_x^{\alpha}K(x, s, C^T Q_n(s), C^T P_n(s))ds. \tag{5.5}$$

By the following grids points: $x_i = \frac{2i+1}{2(n+1)}$, $i = 0, 1, 2, \ldots, n-1$, we collocate (5.5) to obtain the following nonlinear algebraic system:

$$C^T A = D^T \tag{5.6}$$

such that

$$A = \begin{pmatrix} \psi_0(x_0) & \psi_0(x_1) & \cdots & \psi_0(x_{n-1}) \\ \psi_1(x_0) & \psi_1(x_1) & \cdots & \psi_1(x_{n-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{n-1}(x_0) & \psi_{n-1}(x_1) & \cdots & \psi_{n-1}(x_{n-1}) \end{pmatrix}$$

and

$$D^T = [d_0, d_1, \cdots, d_{n-1}]$$

where, for $i = 0, \cdots, n-1$:

$$d_i = \mathcal{D}^{\alpha}g(x_i) + \int_0^1 \mathcal{D}_x^{\alpha}K(x_i, s, C^T Q_n(s), C^T P_n(s))ds.$$

We use the successive approximation method of Picard to solve the system (5.6), then substitute the coefficients of C^T in the formula (5.4) to get the numerical solution of our equation (5.1).

6. Examples

In this section, we suggest several examples to show the efficacy of our proposed method, Let's define the error function as:

$$E_n = \max_{i=0,n-1} |u_n(x_i) - u(x_i)|$$

where $u(x)$ is the exact solution, $u_n(x)$ is the numerical solution, and n represents the degree of Hermite Wavelets polynomial.

6.1. Example 1

Consider the following equation:

$$\begin{cases} u(x) = g(x) + \int_0^1 ln\left[\frac{6}{5}(\cos(s) - xe^{-3s}) + \frac{2}{5}u(s) - \mathcal{D}^\alpha u(s)\right] ds, \quad \forall x \in [0,1], \\ u(0) = 0, \end{cases}$$

and

$$g(x) = \cos(x) + ln\left[\frac{6}{5}(1+x)\right] - \frac{3}{2}.$$

The exact solution of this equation is $u(x) = \sin(x)$, and the fractional order of derivation $\alpha = 0.5$.

6.2. Example 2

Let the Fredholm integro-differential equation:

$$\begin{cases} u(x) = g(x) - \int_0^1 \frac{sin(x+s)}{1+2su(s)+\mathcal{D}^\alpha u(s)} ds, \quad \forall x \in [0,1], \\ u(0) = 0, \end{cases}$$

with $g(x) = e^{-x} - 1 + \cos(1+x) - \cos(x)$ such that the exact solution is $u(x) = e^{-x} - 1$, and the fractional order of derivation $\alpha = 0.75$

6.3. Example 3

Consider the following integro-differential equation:

$$\begin{cases} u(x) = -\frac{5}{2}x^2 + \int_0^1 x^2\sqrt{12 - 3e^{-s} + 2\mathcal{D}^\alpha u(s) + u(s)}ds, \quad \forall x \in [0,1], \\ u(0) = 0, \end{cases}$$

The exact solution of equation is $u(x) = x^2$, and order of derivation is $\alpha = \frac{2}{3}$.

n	E_n
3	6.9888e-05
4	1.4336e-06
5	4.9340e-08
6	6.7237e-10
7	5.4441e-10

Table 6.1: Numerical results of example 6.1.

n	E_n
3	7.9499e-04
4	2.7590e-05
5	7.6178e-07
6	1.7419e-08
7	8.5031e-09

Table 6.2: Numerical results of example 6.2.

n	E_n
3	4.4024e-04
4	5.3991e-05
5	5.7305e-06
6	4.6619e-07
7	4.3553e-08

Table 6.3: Numerical results of example 6.3.

n	E_n
3	1.3017e-05
4	1.7626e-07
5	1.0658e-08
6	7.0663e-11
7	4.7379e-11

Table 6.4: Numerical results of example 6.4.

6.4. Example 4

Let the following equation:

$$\begin{cases} u(x) = e^{-\frac{x}{3}} - \cos(x) + \int_0^1 \sin\left(x - s + \frac{2}{9}s^2 - \frac{4}{3}x + e^{\frac{s}{3}}u(s) + e^{\frac{s}{3}}\mathcal{D}^{\alpha}u(s)\right) ds, \\ \forall x \in [0,1], \\ u(0) = 0, \end{cases}$$

its exact solution is $u(x) = e^{-\frac{x}{3}} - \cos(x)$, and the fractional order for this example is $\alpha = 0.25$.

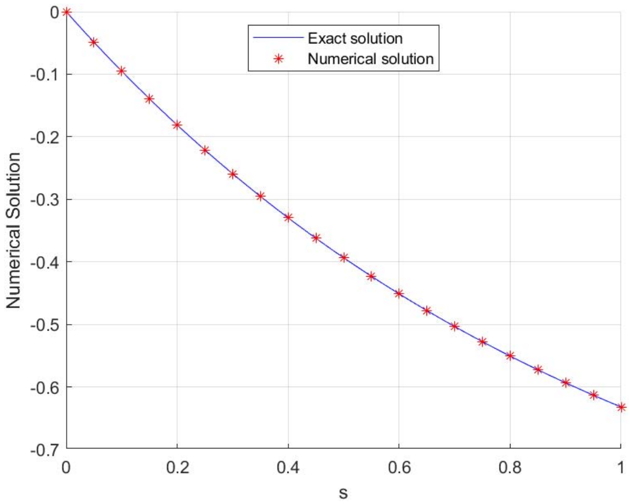

Figure 1: $u(x)$ versus $u_n(x)$ for example 6.1, with $n = 7$.

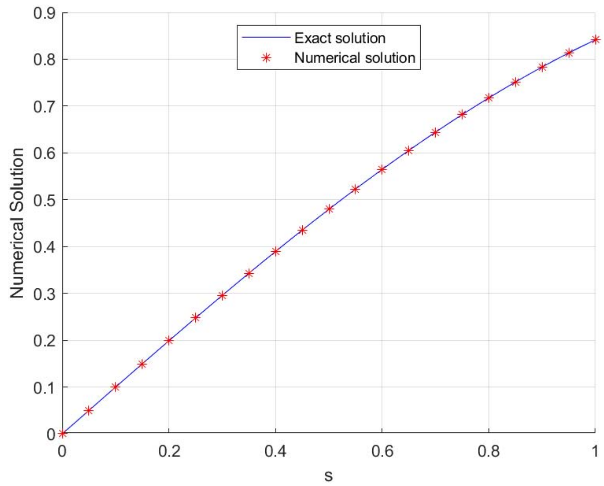

Figure 2: $u(x)$ versus $u_n(x)$ for example 6.2, with $n = 7$.

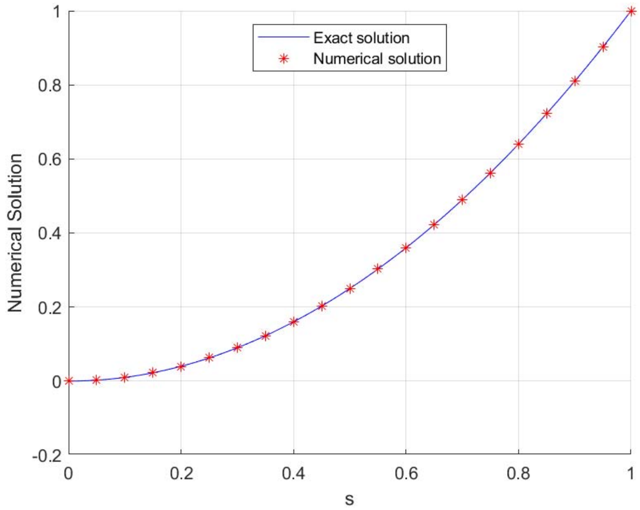

Figure 3: $u(x)$ **versus** $u_n(x)$ **for example 6.3, with** $n = 7$**.**

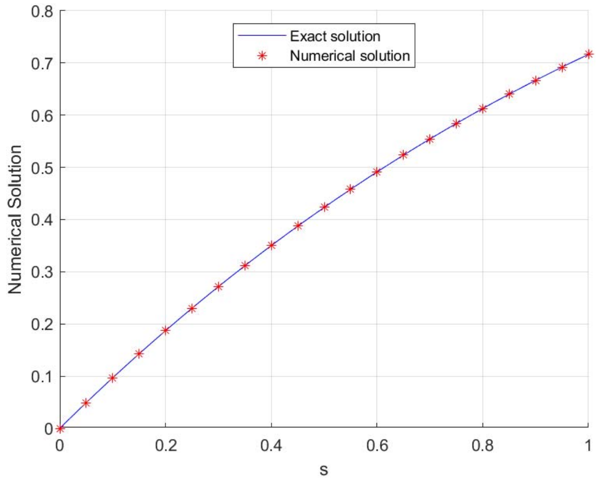

Figure 4: $u(x)$ **versus** $u_n(x)$ **for example 6.4, with** $n = 7$**.**

7. Interpretation of results

Tables 6.1, 6.2, 6.3, and 6.4 represent the error function E_n for various values of the degree n, which affirm that the present method is better when the degree n is larger. As well as, the Figures 1, 2, 3, and 4 show graphical representations of the exact solution and the approximate solution at the same time, which appear to be nearly identical. So the previous proposed examples confirm the efficiency and validity of our numerical method.

8. Conclusion

We have applied the Hermite-Wavelets-Collocation method to approach the solution of nonlinear Fredholm integro-differential equation with the fractional Caputo-Fabrizio derivative. The present method has permitted us to reduce the equation into a nonlinear algebraic system which has been solved by the Picard iterative method. Moreover, the suggested examples have proved the accuracy and validity of this method.

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Youcef Henka
Laboratoire de Mathématiques Appliquées et de Modélisation,
Université 8 Mai 1945 Guelma
B. P. 401 Guelma 24000
Algérie
e-mail: henka.youcef.henka@gmail.com
henka.youcef@univ-guelma.dz

Samir Lemita
Ecole Normale Supérieure-Ouargla,
Cité Ennacer, Ouargla 30000
Algérie
e-mail: lem.samir@gmail.com
Corresponding author

and

Mohamed Zine Aissaoui
Laboratoire de Mathématiques Appliquées et de Modélisation,
Université 8 Mai 1945 Guelma
B. P. 401 Guelma 24000
Algérie
e-mail: aissaouizine@gmail.com